HYPERCONTRACTIONS AND FACTORIZATIONS OF MULTIPLIERS IN ONE AND SEVERAL VARIABLES

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Abstract. We introduce the notion of characteristic functions for commuting tuples of hypercontractions on Hilbert spaces, as a generalization of the notion of Sz.-Nagy and Foias characteristic functions of contractions. We present an explicit method to compute characteristic functions of hypercontractions and relate characteristic functions by means of the factors of Schur-Agler class of functions and universal multipliers on the unit ball in $\mathbb{C}^n$. We also offer some factorization properties of multipliers. Characteristic functions of hypercontrctions are complete unitary invariant. The Drury-Arveson space and the weighted Bergman spaces on the unit ball continues to play a significant role in our consideration. Our results are new even in the special case of single hypercontractions.

1. Introduction

One of the important aspects of the classical Sz.-Nagy and Foias theory [15] is that in order to understand non-self adjoint bounded linear operators on Hilbert spaces, one should also study (analytic) function theory. For instance, if T is a pure contraction on a Hilbert space $\mathcal{H}$ (that is, $\|Th\|_{\mathcal{H}} \leq \|h\|_{\mathcal{H}}$ and $\|T^{*q}h\| \to 0$ as $q \to \infty$ and for all $h \in \mathcal{H}$), then there exist a (coefficient) Hilbert space $\mathcal{E}$ and an M_z^*-invariant closed subspace $\mathcal{Q}$ (*model space*) of $H^2_{\mathcal{E}}(\mathbb{D})$ such that T^* and $M_z^*|_{\mathcal{Q}}$ are unitarily equivalent. Here M_z is the multiplication operator by the coordinate function z (or, *shift*) on the $\mathcal{E}$*-valued Hardy space* $H^2_{\mathcal{E}}(\mathbb{D})$ over the open unit disc $\mathbb{D}$. Moreover, $\mathcal{Q}$ is uniquely determined by the *characteristic function* of T in an appropriate sense. The Sz.-Nagy and Foias characteristic function of a contraction is a canonical operator-valued analytic function on $\mathbb{D}$ and a complete unitary invariant.

This says, on the one hand, pure contractions on Hilbert spaces dilates to shifts on vector-valued Hardy spaces over the unit disc, and on the other hand, the model spaces (as Hilbert subspaces of vector-valued Hardy spaces) are explicitly and uniquely determined by characteristic functions.

In this context, it should be remembered that the concept of Sz.-Nagy and Foias "dilations and analytic model theory", as above, is most useful in operator theory having important applications in various fields. This has had an enormous influence on the development of operator theory and function theory in one and several variables. Needless to say, one goal of multivariable operator theory and function theory of several complex variables is to examine

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whether commuting tuples of contractions on Hilbert spaces admit analytic models as nice as Sz.-Nagy and Foias analytic models for contractions.

Following Sz.-Nagy and Foias, Agler, in his seminal papers [1, 2], introduced a dilation theory for m-hypercontractions: A pure m-hypercontraction dilates to shift on a vector-valued m-weighted Bergman space over the unit disc in $\mathbb{C}$. Agler's idea was further generalized by Muller and Vasilescu [14] to commuting tuples of operators: A pure n-tuple of m-hypercontraction dilates to n-shifts on a vector-valued m-weighted Bergman space over the unit ball in $\mathbb{C}^n$ (see Section 2 for more details).

This paper concerns a complete unitary invariant, namely characteristic functions, one of the basic questions which center around the Agler, and Muller and Vasilescu's dilation theory, of commuting tuples of pure m-hypercontractions on Hilbert spaces.

The problem of characteristic functions for hypercontractions and wandering subspaces of shift invariant subspaces of weighted Bergman spaces in one-variable goes back to Olofsson [16, 17] (also see Ball and Bolotnikov [6]). Then in [10], Eschmeier examined Olofsson's approach in several variables (also see Popescu [19]). However, Eschmeier's approach to characteristic functions appears to be more abstract than the familiar characteristic functions of single contractions or row contractions [9].

Here we take a completely different approach to this problem. Namely, to each pure m-hypercontraction on a Hilbert space, we associate a canonical triple consisting of a Hilbert space and two bounded linear operators, and refer to this triple as a characteristic triple of the pure m-hypercontraction. The characteristic function of a pure m-hypercontraction, completely determined by a characteristic triple, is an operator-valued analytic function on the open unit ball in $\mathbb{C}^n$. Characteristic triple of a pure m-hypercontraction is unique up to unitary equivalence (in an appropriate sense), which also yields that the characteristic function is a complete unitary invariant. We prove that the joint invariant subspaces of a pure m-hypercontraction is completely determined by the factors of the characteristic function. Unlike the case of 1-hypercontractions (or row contractions) [9], the characteristic function of a pure m-hypercontraction does not admit a transfer function realization. However, we prove that the characteristic function of pure m-hypercontraction can be (canonically) represented as a product of a universal multiplier and a transfer function (or a Drury-Arveson multiplier). This result is a byproduct of a general factorization theorem for contractive multipliers from vector-valued Drury-Arveson spaces to a class of reproducing kernel Hilbert spaces on $\mathbb{B}^n$. The general factorization theorem for contractive multipliers also yields parametrizations of wandering subspaces of the joint shift invariant subspaces of reproducing kernel Hilbert spaces.

The results and the method we introduce here seems to be new even in the single hypercontractions case.

We now describe our main results more precisely. Let m and n be natural numbers, $\mathbb{Z}_+^n$ be the set of n-tuples of non-negative integers, that is

$$\mathbb{Z}_+^n = \{\boldsymbol{k} = (k_1, \dots, k_n) : k_1, \dots, k_n \in \mathbb{Z}_+\},$$

and let

$$\mathbb{B}^n = \{\boldsymbol{z} = (z_1, \dots, z_n) \in \mathbb{C}^n : \sum_{i=1}^n |z_i|^2 < 1\},$$

the open unit ball in $\mathbb{C}^n$. We denote by $\mathcal{H}$, $\mathcal{K}$, $\mathcal{E}$ etc. as separable Hilbert spaces over $\mathbb{C}$, and by $\mathcal{B}(\mathcal{H})$ the set of all bounded linear operators on $\mathcal{H}$.
Unless otherwise stated, T will always mean a commuting n-tuple of bounded linear operators $\{T_1, \ldots, T_n\}$ on some Hilbert space $\mathcal{H}$. We also adopt the following notations:

$$T^{\boldsymbol{k}} = T_1^{k_1} \cdots T_n^{k_n} \quad \text{and} \quad T^{*\boldsymbol{k}} = T_1^{*k_1} \cdots T_n^{*k_n},$$

and the multinomial coefficient $\rho_m(\boldsymbol{k})$ as

$$\rho_m(\boldsymbol{k}) = \frac{(m + |\boldsymbol{k}| - 1)!}{\boldsymbol{k}!(m-1)!}, \tag{1.1}$$

and

$$\rho_0(\boldsymbol{k}) = \begin{cases} 1 & \text{if } \boldsymbol{k} = \boldsymbol{0} \\ 0 & \text{otherwise}, \end{cases} \tag{1.2}$$

for all $\boldsymbol{k} \in \mathbb{Z}_+^n$. We say that T is a *row contraction* if the row operator $(T_1, \ldots, T_n) : \mathcal{H}^n \to \mathcal{H}$, denoted again by T and defined by

$$T(h_1, \ldots, h_n) = \sum_{i=1}^{n} T_i h_i \qquad (h_i \in \mathcal{H}),$$

is a contraction. More generally, if we define the completely positive map $\sigma_T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by

$$\sigma_T(X) = \sum_{i=1}^{n} T_i X T_i^* \qquad (X \in \mathcal{B}(\mathcal{H})),$$

then T is said to be an *m-hypercontraction* if

$$(I_{\mathcal{B}(\mathcal{H})} - \sigma_T)^p (I_{\mathcal{H}}) \geq 0,$$

for $p = 1, m$. Note that T is an 1-contraction if and only if T is a *row contraction* (cf. [5]). It is now immediate that

$$(I_{\mathcal{B}(\mathcal{H})} - \sigma_T)^p (I_{\mathcal{H}}) = \sum_{j=0}^{p} (-1)^j \binom{p}{j} \sum_{|\boldsymbol{k}|=j} \rho_1(\boldsymbol{k}) T^{\boldsymbol{k}} T^{*\boldsymbol{k}}. \tag{1.3}$$

With this notation we get the following interpretation of hypercontractions: T is an m-hypercontraction if and only if T is a row contraction (that is, $(I_{\mathcal{B}(\mathcal{H})} - \sigma_T)(I_{\mathcal{H}}) \geq 0$) and $(I_{\mathcal{B}(\mathcal{H})} - \sigma_T)^m (I_{\mathcal{H}}) \geq 0$. For each m-hypercontraction T on $\mathcal{H}$, we set the *defect operator* D_{m,T^*} as

$$D_{m,T^*} = \left[(I_{\mathcal{B}(\mathcal{H})} - \sigma_T)^m (I_{\mathcal{H}}) \right]^{\frac{1}{2}},$$

and the *defect space* $\mathcal{D}_{m,T^*}$ as

$$\mathcal{D}_{m,T^*} = \overline{\operatorname{ran}} D_{m,T^*}.$$

An m-hypercontraction T is said to be *pure* (cf. [10, 14, 19]) if the strong operator limit of $\sigma_T^p(I_{\mathcal{H}})$ is 0 as $p \to \infty$.

Now let $T = (T_1, \ldots, T_n)$ be a commuting n-tuple of pure m-hypercontraction on a Hilbert space $\mathcal{H}$. After reviewing the basic definitions and results of the theory of m-hypercontractions in Section 2, we prove the existence of a canonical contraction $C_{m,T} : \mathcal{H} \to l^2(\mathbb{Z}_+^n, \mathcal{D}_{m,T^*})$, a Hilbert space $\mathcal{E}$, and bounded linear operators $B \in \mathcal{B}(\mathcal{E}, \mathcal{H}^n)$ and $D \in \mathcal{B}(\mathcal{E}, l^2(\mathbb{Z}_+^n, \mathcal{D}_{m,T^*}))$ such that the operator matrix

$$U = \begin{bmatrix} T^* & B \\ C_{m,T} & D \end{bmatrix} : \mathcal{H} \oplus \mathcal{E} \to \mathcal{H}^n \oplus l^2(\mathbb{Z}_+^n, \mathcal{D}_{m,T^*}),$$

is unitary (see Theorem 2.1).
The triple $(\mathcal{E}, B, D)$ is referred to as a *characteristic triple* of T. The *characteristic function* of T corresponding to the triple $(\mathcal{E}, B, D)$ is the $\mathcal{B}(\mathcal{E}, \mathcal{D}_{m,T^*})$-valued analytic function

$$\Phi_T : \mathbb{B}^n \to \mathcal{B}(\mathcal{E}, \mathcal{D}_{m,T^*}),$$

defined by

$$\Phi_T(z) = \sum_{\boldsymbol{k} \in \mathbb{Z}_+^n} \sqrt{\rho_{m-1}(\boldsymbol{k})} D_{\boldsymbol{k}} z^{\boldsymbol{k}} + D_{m,T^*}(I_{\mathcal{H}} - ZT^*)^{-m} ZB \qquad \big(\boldsymbol{z} \in \mathbb{B}^n\big),$$

where $Z = (z_1 I_{\mathcal{H}}, \ldots, z_n I_{\mathcal{H}})$ for all $\boldsymbol{z} \in \mathbb{B}^n$, $D_{\boldsymbol{k}}$, $\boldsymbol{k} \in \mathbb{Z}_+^n$, is the $\boldsymbol{k}$-th entry of the "column matrix" D (see (3.1) for more details).

The operator-valued analytic function Φ_T may be viewed as a counterpart of Sz.-Nagy and Foias characteristic functions for contractions. Indeed, in Theorem 3.1 in Section 3, we prove that Φ_T defines a partially isometric multiplier from $H_n^2(\mathcal{E})$, the $\mathcal{E}$-valued Drury-Arveson space over the open unit ball $\mathbb{B}^n$ [5], to $\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})$, the $\mathcal{D}_{m,T^*}$-valued weighted Bergman space over $\mathbb{B}^n$. Moreover,

$$\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*}) \ominus \Phi_T \, H_n^2(\mathcal{E}),$$

is the model space of the pure m-hypercontraction T in the sense of Muller and Vasilescu [14].

Section 4 deals with universal multipliers corresponding to weight sequences and parameterizations of wandering subspaces of commuting tuples of shift operators. In Theorem 4.2 we prove that any multiplier from a vector-valued Drury-Arveson space to a (class of) vector-valued reproducing kernel Hilbert space on $\mathbb{B}^n$ can be factored as a product of a universal multiplier (which depends only on the kernel function and coefficient Hilbert space) and a Schur-Agler class of functions. We also point out that the unique factorization property holds in the setting of "inner" functions in several variables (see Theorem 4.3). Then, in Section 5, we turn to a canonical factorization of Φ_T. Recall that [3] given Hilbert spaces $\mathcal{E}$ and $\mathcal{F}$ and an analytic function $\Theta : \mathbb{B}^n \to \mathcal{B}(\mathcal{E}, \mathcal{F})$, Θ is a contractive multiplier from $H_n^2(\mathcal{E})$ to $H_n^2(\mathcal{F})$ if and only if there exist auxiliary Hilbert space $\mathcal{H}$ and a unitary

$$W : \mathcal{H} \oplus \mathcal{E} \to \Big(\bigoplus_{i=1}^n \mathcal{H} \Big) \oplus \mathcal{F},$$

such that, writing W as

$$W = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

one has the following *transfer function* realization (cf. [3])

$$\Theta(\boldsymbol{z}) = D + C(I_{\mathcal{H}} - ZA)^{-1}ZB \qquad (\boldsymbol{z} \in \mathbb{B}^n).$$

In Theorem 5.1, we prove that Φ_T factors through a (canonical) transfer function. More specifically

$$\Phi_T(\boldsymbol{z}) = \Psi_{\beta(m),\mathcal{D}_{m,T^*}}(\boldsymbol{z})\tilde{\Phi}_T(\boldsymbol{z}),$$

where

$$\tilde{\Phi}_T(\boldsymbol{z}) = D + C_{m,T}(I_{\mathcal{H}} - ZT^*)^{-1}ZB,$$

is the transfer function of the unitary matrix U corresponding to $(\mathcal{E}, B, D)$, and

$$\Psi_{\beta(m),\mathcal{D}_{m,T^*}}(\boldsymbol{z}) = \begin{cases} \begin{bmatrix} \cdots & \sqrt{\rho_{m-1}(\boldsymbol{k})}z^{\boldsymbol{k}} I_{\mathcal{D}_{m,T^*}} & \cdots \end{bmatrix}_{\boldsymbol{k}\in\mathbb{Z}_+^n} & \text{if } m \geq 2 \\ \begin{bmatrix} I_{\mathcal{D}_{1,T^*}} & 0 & 0 & \cdots \end{bmatrix} & \text{if } m = 1, \end{cases}$$

for all $\boldsymbol{z} \in \mathbb{B}^n$. Here $\Psi_{\beta(m),\mathcal{D}_{m,T^*}}$ is the universal multiplier from $H^2_n(l^2(\mathbb{Z}_+^n, \mathcal{D}_{m,T^*}))$ to $\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})$. In the final section, Section 6, we link up our results with characteristic functions of pure row contractions [9].

2. Preliminaries and Characteristic triples

We begin by exploring natural examples of pure m-hypercontractions. Let p be a natural number, and let

$$K_p(\boldsymbol{z}, \boldsymbol{w}) = (1 - \sum_{i=1}^{n} z_i \bar{w}_i)^{-p} \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n).$$

Then K_p is a positive-definite kernel on $\mathbb{B}^n$. Denote by $\mathbb{H}_p$ the reproducing kernel Hilbert space (of scalar-valued analytic functions on $\mathbb{B}^n$) corresponding to the kernel K_p. If $\boldsymbol{w} \in \mathbb{B}^n$, then we let $K_p(\cdot, \boldsymbol{w})$ denote the function in $\mathbb{H}_p$ defined by

$$(K_p(\cdot, \boldsymbol{w}))(\boldsymbol{z}) = K_p(\boldsymbol{z}, \boldsymbol{w}) \qquad (\boldsymbol{z} \in \mathbb{B}^n).$$

Given a Hilbert space $\mathcal{E}$, we denote by $\mathbb{H}_p(\mathbb{B}^n, \mathcal{E})$ the reproducing kernel Hilbert space corresponding to the $\mathcal{B}(\mathcal{E})$-valued kernel

$$(\boldsymbol{z}, \boldsymbol{w}) \mapsto K_p(\boldsymbol{z}, \boldsymbol{w})I_{\mathcal{E}},$$

on $\mathbb{B}^n$. We simply write $\mathbb{H}_p$ instead of $\mathbb{H}_p(\mathbb{B}^n, \mathbb{C})$ if $\mathcal{E} = \mathbb{C}$. Note that for $\boldsymbol{z} \in \mathbb{B}^n$, we have (cf. page 983, [14])

$$(1 - \sum_{i=1}^{n} z_i)^{-p} = \sum_{\boldsymbol{k}\in\mathbb{Z}_+^n} \rho_p(\boldsymbol{k}) z^{\boldsymbol{k}},$$

where $z^{\boldsymbol{k}} = z_1^{k_1} \cdots z_n^{k_n}$ for all $\boldsymbol{k} \in \mathbb{Z}_+^n$. Then

$$\mathbb{H}_p(\mathbb{B}^n, \mathcal{E}) = \{f = \sum_{\boldsymbol{k}\in\mathbb{Z}_+^n} a_{\boldsymbol{k}} z^{\boldsymbol{k}} \in \mathcal{O}(\mathbb{B}^n, \mathcal{E}) : \|f\|^2 := \sum_{\boldsymbol{k}\in\mathbb{Z}_+^n} \frac{\|a_{\boldsymbol{k}}\|^2_{\mathcal{E}}}{\rho_p(\boldsymbol{k})} < \infty\}.$$

In particular, $\mathbb{H}_1(\mathbb{B}^n,\mathcal{E})$, $\mathbb{H}_n(\mathbb{B}^n,\mathcal{E})$ and $\mathbb{H}_{n+1}(\mathbb{B}^n,\mathcal{E})$ represents the well-known $\mathcal{E}$-valued *Drury-Arveson space*, the Hardy space and the Bergman space over $\mathbb{B}^n$, respectively. Moreover, for each $p > n+1$, $\mathbb{H}_p(\mathbb{B}^n,\mathcal{E})$ is an $\mathcal{E}$-valued weighted Bergman space over $\mathbb{B}^n$ (cf. [23]). Following standard notation, we denote the Drury-Arveson space $\mathbb{H}_1(\mathbb{B}^n,\mathcal{E})$ by $H^2_n(\mathcal{E})$. Again, if $\mathcal{E}=\mathbb{C}$, then we simply denote $H^2_n(\mathbb{C})$ by H^2_n.
An easy computation shows that $(M_{z_1},\dots,M_{z_n})$, the commuting tuple of multiplication operators by the coordinate functions $\{z_1,\dots,z_n\}$, defines a pure m-hypercontraction on $\mathbb{H}_p(\mathbb{B}^n,\mathcal{E})$ for all $m\le p$.
To simplify the notation, we often identify $\mathbb{H}_p\otimes\mathcal{E}$ with $\mathbb{H}_p(\mathbb{B}^n,\mathcal{E})$ via the unitary map defined by $\boldsymbol{z}^{\boldsymbol{k}}\otimes\eta\mapsto\boldsymbol{z}^{\boldsymbol{k}}\eta$, for all $\boldsymbol{k}\in\mathbb{Z}^n_+$ and $\eta\in\mathcal{E}$. As a consequence, we can identify $M_z\otimes I_{\mathcal{E}}$ on $\mathbb{H}_p\otimes\mathcal{E}$ with M_z on $\mathbb{H}_p(\mathbb{B}^n,\mathcal{E})$ (as tuples of operators).

Recall that a holomorphic map $\Phi:\mathbb{B}^n\to\mathcal{B}(\mathcal{E}_1,\mathcal{E}_2)$, for some Hilbert spaces $\mathcal{E}_1$ and $\mathcal{E}_2$, is said to be a *multiplier* from $H^2_n(\mathcal{E}_1)$ to $\mathbb{H}_m(\mathbb{B}^n,\mathcal{E}_2)$ if

$$\Phi f\in\mathbb{H}_m(\mathbb{B}^n,\mathcal{E}_2),$$

for all $f\in H^2_n(\mathcal{E}_1)$. We denote by $\mathcal{M}(H^2_n(\mathcal{E}_1),\mathbb{H}_m(\mathbb{B}^n,\mathcal{E}_2))$ the set of all multipliers from $H^2_n(\mathcal{E}_1)$ to $\mathbb{H}_m(\mathbb{B}^n,\mathcal{E}_2)$. Note also that a multiplier $\Phi\in\mathcal{M}(H^2_n(\mathcal{E}_1),\mathbb{H}_m(\mathbb{B}^n,\mathcal{E}_2))$ gives rise to a bounded linear operator

$$M_\Phi: H^2_n(\mathcal{E}_1)\to\mathbb{H}_m(\mathbb{B}^n,\mathcal{E}_2),\qquad f\mapsto\Phi f,$$

known as the multiplication operator corresponding to Φ. Multipliers can be characterized as follows: Let $X\in\mathcal{B}(H^2_n(\mathcal{E}_1),\mathbb{H}_m(\mathbb{B}^n,\mathcal{E}_2))$. Then $X\in\mathcal{M}(H^2_n(\mathcal{E}_1),\mathbb{H}_m(\mathbb{B}^n,\mathcal{E}_2))$ if and only if

$$X(M_{z_i}\otimes I_{\mathcal{E}_1})=(M_{z_i}\otimes I_{\mathcal{E}_2})X,$$

for all $i=1,\dots,n$. For more details about multipliers on reproducing kernel Hilbert spaces in our present context, we refer to [21].

Finally, recall also that if T is a pure m-hypercontraction on $\mathcal{H}$, then the *canonical dilation map* (see [8], and also see [14]) $\Pi_m:\mathcal{H}\to\mathbb{H}_m(\mathbb{B}^n,\mathcal{D}_{m,T^*})$, defined by

$$(\Pi_m h)(\boldsymbol{z})=D_{m,T^*}(I_{\mathcal{H}}-ZT^*)^{-m}h\qquad(h\in\mathcal{H},\boldsymbol{z}\in\mathbb{B}^n),\tag{2.1}$$

is an isometry and

$$\Pi_m T_i^*=M_{z_i}^*\Pi_m,$$

for all $i=1,\dots,n$, where $Z:\mathcal{H}^n\to\mathcal{H}$ is the row contraction $Z=(z_1I_{\mathcal{H}},\dots,z_nI_{\mathcal{H}})$, $\boldsymbol{z}\in\mathbb{B}^n$. In particular, if

$$\mathcal{Q}_{m,T}=\Pi_m\mathcal{H},$$

then $\mathcal{Q}_{m,T}$, the *canonical model space* corresponding to T, is a joint $(M^*_{z_1},\dots,M^*_{z_n})$-invariant subspace and $(P_{\mathcal{Q}_{m,T}}M_{z_1}|_{\mathcal{Q}_{m,T}},\dots,P_{\mathcal{Q}_{m,T}}M_{z_1}|_{\mathcal{Q}_{m,T}})$ on $\mathcal{Q}_{m,T}$ and $(T_1,\dots,T_n)$ on $\mathcal{H}$ are unitarily equivalent (see [14, 21]). This shows, in particular, that pure m-hypercontractions on Hilbert spaces are precisely (in the sense of unitary equivalence) the compressions of M_z to joint co-invariant subspaces of vector-valued $\mathbb{H}_m$-spaces.
On the other hand, $\mathcal{S}_{m,T}$, the *canonical invariant subspace* corresponding to T, defined by

$$\mathcal{S}_{m,T}=\mathbb{H}_m(\mathbb{B}^n,\mathcal{D}_{m,T^*})\ominus\Pi_m\mathcal{H},$$

is a joint $(M_{z_1}, \ldots, M_{z_n})$-invariant subspace of $\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})$, and hence by a Beurling-Lax-Halmos type theorem (see [21, Theorem 4.4]) it follows that

$$\mathcal{S}_{m,T} = \Phi H^2_n(\mathcal{E}), \tag{2.2}$$

for some Hilbert space $\mathcal{E}$ and a partially isometric multiplier $\Phi \in \mathcal{M}(H^2_n(\mathcal{E}), \mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*}))$.

We turn now to the main content of this section. Let T be a pure m-hypercontraction on $\mathcal{H}$. Since

$$\langle \boldsymbol{z}^{\boldsymbol{k}}, \boldsymbol{z}^{\boldsymbol{l}} \rangle_{\mathbb{H}_m} = \frac{1}{\rho_m(\boldsymbol{k})} \delta_{\boldsymbol{k},\boldsymbol{l}},$$

for all $\boldsymbol{k}, \boldsymbol{l} \in \mathbb{Z}^n_+$, and the canonical dilation map Π_m is an isometry, that is, $\Pi_m^* \Pi_m = I_{\mathcal{H}}$, and

$$(\Pi_m h)(\boldsymbol{z}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^n_+} \rho_m(\boldsymbol{k}) (D_{m,T^*} T^{*\boldsymbol{k}} h) \boldsymbol{z}^{\boldsymbol{k}} \qquad (\boldsymbol{z} \in \mathbb{B}^n, h \in \mathcal{H}),$$

it follows that

$$\sum_{\boldsymbol{k} \in \mathbb{Z}^n_+} \rho_m(\boldsymbol{k}) T^{\boldsymbol{k}} D^2_{m,T^*} T^{*\boldsymbol{k}} = I_{\mathcal{H}}. \tag{2.3}$$

Moreover, since

$$\rho_0(\boldsymbol{k}) = \begin{cases} 1 & \text{if } \boldsymbol{k} = \boldsymbol{0} \\ 0 & \text{otherwise,} \end{cases}$$

an easy computation shows that (cf. page 96, [10])

$$\rho_m(\boldsymbol{k}) = \rho_{m-1}(\boldsymbol{k}) + \sum_{\substack{i=1 \\ k_i \geq 1}}^{n} \rho_m(\boldsymbol{k} - \boldsymbol{e}_i), \tag{2.4}$$

where

$$\boldsymbol{k} - \boldsymbol{e}_i = \begin{cases} (k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_n) & \text{if } k_i \geq 1 \\ 0 & \text{if } k_i = 0, \end{cases}$$

and $\boldsymbol{k} \in \mathbb{Z}^n_+$. Hence, by (2.3) we have

$$\sum_{\boldsymbol{k} \in \mathbb{Z}^n_+} \rho_{m-1}(\boldsymbol{k}) T^{\boldsymbol{k}} D^2_{m,T^*} T^{*\boldsymbol{k}} \leq \sum_{\boldsymbol{k} \in \mathbb{Z}^n_+} \rho_m(\boldsymbol{k}) T^{\boldsymbol{k}} D^2_{m,T^*} T^{*\boldsymbol{k}} = I_{\mathcal{H}}.$$

Then the linear map $C_{m,T} : \mathcal{H} \to l^2(\mathbb{Z}^n_+, \mathcal{D}_{m,T^*})$ defined by

$$C_{m,T}(h) = (\sqrt{\rho_{m-1}(\boldsymbol{k})} D_{m,T^*} T^{*\boldsymbol{k}} h)_{\boldsymbol{k} \in \mathbb{Z}^n_+} \qquad (h \in \mathcal{H}), \tag{2.5}$$

is a contraction. It is often convenient to represent $C_{m,T}$ as the column matrix

$$C_{m,T} = \begin{bmatrix} \vdots \\ \sqrt{\rho_{m-1}(\boldsymbol{k})} D_{m,T^*} T^{*\boldsymbol{k}} \\ \vdots \end{bmatrix}_{\boldsymbol{k} \in \mathbb{Z}^n_+}. \tag{2.6}$$

Now using (2.3) twice, we have

$$
\begin{aligned}
I_{\mathcal{H}} - C_{m,T}^* C_{m,T} &= \sum_{\boldsymbol{k}\in\mathbb{Z}_+^n} \rho_m(\boldsymbol{k}) T^{\boldsymbol{k}} D_{m,T^*}^2 T^{*\boldsymbol{k}} - \sum_{\boldsymbol{k}\in\mathbb{Z}_+^n} \rho_{m-1}(\boldsymbol{k}) T^{\boldsymbol{k}} D_{m,T^*}^2 T^{*\boldsymbol{k}} \\
&= \sum_{\boldsymbol{k}\in\mathbb{Z}_+^n} \Big(\rho_m(\boldsymbol{k}) - \rho_{m-1}(\boldsymbol{k})\Big) T^{\boldsymbol{k}} D_{m,T^*}^2 T^{*\boldsymbol{k}} \\
&= \sum_{\boldsymbol{k}\in\mathbb{Z}_+^n} \Big(\sum_{\substack{i=1\\ k_i\geq 1}}^n \rho_m(\boldsymbol{k}-\boldsymbol{e}_i)\Big) T^{\boldsymbol{k}} D_{m,T^*}^2 T^{*\boldsymbol{k}} \\
&= \sum_{i=1}^n \sum_{\boldsymbol{k}\in\mathbb{Z}_+^n} \rho_m(\boldsymbol{k}) T^{\boldsymbol{k}+\boldsymbol{e}_i} D_{m,T^*}^2 T^{*(\boldsymbol{k}+\boldsymbol{e}_i)} \\
&= \sum_{i=1}^n T_i \Big(\sum_{\boldsymbol{k}\in\mathbb{Z}_+^n} \rho_m(\boldsymbol{k}) T^{\boldsymbol{k}} D_{m,T^*}^2 T^{*\boldsymbol{k}}\Big) T_i^* \\
&= \sum_{i=1}^n T_i T_i^*,
\end{aligned}
$$

that is

$$
TT^* + C_{m,T}^* C_{m,T} = I_{\mathcal{H}}, \tag{2.7}
$$

and therefore

$$
X_T = \begin{bmatrix} T^* \\ C_{m,T} \end{bmatrix} : \mathcal{H} \to \mathcal{H}^n \oplus l^2(\mathbb{Z}_+^n, \mathcal{D}_{m,T^*}),
$$

is an isometry. By adding a suitable Hilbert space $\mathcal{E}$, we extend X_T on $\mathcal{H}$ to a unitary $U : \mathcal{H} \oplus \mathcal{E} \to \mathcal{H}^n \oplus l^2(\mathbb{Z}_+^n, \mathcal{D}_{m,T^*})$, and set

$$
U := \begin{bmatrix} X_T & Y_T \end{bmatrix} : \mathcal{H} \oplus \mathcal{E} \to \mathcal{H}^n \oplus l^2(\mathbb{Z}_+^n, \mathcal{D}_{m,T^*}),
$$

where $Y_T = U|_{\mathcal{E}} : \mathcal{E} \to \mathcal{H}^n \oplus l^2(\mathbb{Z}_+^n, \mathcal{D}_{m,T^*})$. If we set $Y_T = \begin{bmatrix} B \\ D \end{bmatrix}$ where $B = P_{\mathcal{H}^n} Y_T \in \mathcal{B}(\mathcal{E}, \mathcal{H}^n)$ and $D = P_{l^2(\mathbb{Z}_+^n, \mathcal{D}_{m,T^*})} Y_T \in \mathcal{B}(\mathcal{E}, l^2(\mathbb{Z}_+^n, \mathcal{D}_{m,T^*}))$, then

$$
U = \begin{bmatrix} T^* & B \\ C_{m,T} & D \end{bmatrix} : \mathcal{H} \oplus \mathcal{E} \to \mathcal{H}^n \oplus l^2(\mathbb{Z}_+^n, \mathcal{D}_{m,T^*}).
$$

Summarizing, we have the following result.

Theorem 2.1. *Let T be a pure m-hypercontraction on $\mathcal{H}$. Then the map $C_{m,T} : \mathcal{H} \to l^2(\mathbb{Z}_+^n, \mathcal{D}_{m,T^*})$ defined by*

$$
C_{m,T}(h) = \Big(\sqrt{\rho_{m-1}(\boldsymbol{k})} D_{m,T^*} T^{*\boldsymbol{k}} h\Big)_{\boldsymbol{k}\in\mathbb{Z}_+^n} \qquad (h \in \mathcal{H}),
$$

is a contraction, and there exist a Hilbert space $\mathcal{E}$ and a bounded linear operator

$$Y_T = \begin{bmatrix} B \\ D \end{bmatrix} : \mathcal{E} \to \mathcal{H}^n \oplus l^2(\mathbb{Z}_+^n, \mathcal{D}_{m,T^*}),$$

such that

$$\begin{bmatrix} X_T & Y_T \end{bmatrix} = \begin{bmatrix} T^* & B \\ C_{m,T} & D \end{bmatrix} : \mathcal{H} \oplus \mathcal{E} \to \mathcal{H}^n \oplus l^2(\mathbb{Z}_+^n, \mathcal{D}_{m,T^*}),$$

is a unitary.

This motivates the following definition: Let T be a pure m-hypercontraction on $\mathcal{H}$, $m \geq 1$. A triple $(\mathcal{E}, B, D)$ consisting of a Hilbert space $\mathcal{E}$ and bounded linear operators $B \in \mathcal{B}(\mathcal{E}, \mathcal{H}^n)$ and $D \in \mathcal{B}(\mathcal{E}, l^2(\mathbb{Z}_+^n, \mathcal{D}_{m,T^*}))$ is said to be a *characteristic triple* of T if

$$\begin{bmatrix} X_T & Y_T \end{bmatrix} := \begin{bmatrix} T^* & B \\ C_{m,T} & D \end{bmatrix} : \mathcal{H} \oplus \mathcal{E} \to \mathcal{H}^n \oplus l^2(\mathbb{Z}_+^n, \mathcal{D}_{m,T^*})$$

is a unitary.

Characteristic triple of a pure m-hypercontraction is unique in the following sense:

Theorem 2.2. *Let T be a pure m-hypercontraction on $\mathcal{H}$, and let $(\mathcal{E}, B, D)$ and $(\tilde{\mathcal{E}}, \tilde{B}, \tilde{D})$ be characteristic triples of T. Then there exists a unitary $U : \tilde{\mathcal{E}} \to \mathcal{E}$ such that*

$$(\tilde{\mathcal{E}}, \tilde{B}, \tilde{D}) = (U^* \mathcal{E}, BU, DU).$$

Proof. Since

$$\begin{bmatrix} X_T & Y_T \end{bmatrix} = \begin{bmatrix} T^* & B \\ C_{m,T} & D \end{bmatrix} : \mathcal{H} \oplus \mathcal{E} \to \mathcal{H}^n \oplus l^2(\mathbb{Z}_+^n, \mathcal{D}_{m,T^*}),$$

and

$$\begin{bmatrix} X_T & \tilde{Y}_T \end{bmatrix} = \begin{bmatrix} T^* & \tilde{B} \\ C_{m,T} & \tilde{D} \end{bmatrix} : \mathcal{H} \oplus \tilde{\mathcal{E}} \to \mathcal{H}^n \oplus l^2(\mathbb{Z}_+^n, \mathcal{D}_{m,T^*}),$$

are unitary operators, it follows that $Y_T = \begin{bmatrix} B \\ D \end{bmatrix}$ and $\tilde{Y}_T = \begin{bmatrix} \tilde{B} \\ \tilde{D} \end{bmatrix}$ are isometries and

$$\text{ran} Y_T = \text{ran} \tilde{Y}_T.$$

By Douglas lemma, we have

$$\tilde{Y}_T = Y_T U,$$

for some unitary $U : \tilde{\mathcal{E}} \to \mathcal{E}$, and hence

$$\tilde{B} = BU \text{ and } \tilde{D} = DU.$$

This completes the proof. ■

Characteristic triples of pure m-hypercontractions, $m \geq 1$, will play a key role in what follows. The special case $m = 1$ will be treated in the final section of this paper.

We conclude this section with an explicit construction of a characteristic triple of an m-hypercontraction T on a Hilbert space $\mathcal{H}$: Let X_T be as above. Consider

$$\mathcal{E}_T = \Big(\text{ran} X_T\Big)^{\perp},$$

and the inclusion map

$$i = \begin{bmatrix} B_T \\ D_T \end{bmatrix} : \left(\operatorname{ran} X_T\right)^{\perp} \hookrightarrow \mathcal{H}^n \oplus l^2(\mathbb{Z}_+^n, \mathcal{D}_{m,T^*}).$$

Then it readily follows that $(\mathcal{E}_T, B_T, D_T)$ is a characteristic triple of T.

3. Characteristic Functions

In this section, we continue, by means of operator-valued analytic functions corresponding to characteristic triples, the exploration of pure m-hypercontractions. Here the operator-valued analytic functions will play a similar role as Sz.-Nagy and Foias characteristic functions for contractions.

Let T be a pure m-hypercontraction on $\mathcal{H}$, and let $(\mathcal{E}, B, D)$ be a characteristic triple of T. Note that D can be represented by a column matrix

$$D = \begin{bmatrix} \vdots \\ D_{\boldsymbol{k}} \\ \vdots \end{bmatrix}_{\boldsymbol{k} \in \mathbb{Z}_+^n} : \mathcal{E} \to l^2(\mathbb{Z}_+^n, \mathcal{D}_{m,T^*}), \tag{3.1}$$

where $D_{\boldsymbol{k}} \in \mathcal{B}(\mathcal{E}, \mathcal{D}_{m,T^*})$, $\boldsymbol{k} \in \mathbb{Z}_+^n$. Define

$$\Phi_T : \mathbb{B}^n \to \mathcal{B}(\mathcal{E}, \mathcal{D}_{m,T^*}),$$

by

$$\Phi_T(\boldsymbol{z}) = \Big(\sum_{\boldsymbol{k} \in \mathbb{Z}_+^n} \sqrt{\rho_{m-1}(\boldsymbol{k})} D_{\boldsymbol{k}} z^{\boldsymbol{k}} \Big) + D_{m,T^*}(I_{\mathcal{H}} - ZT^*)^{-m} ZB \qquad (\boldsymbol{z} \in \mathbb{B}^n). \tag{3.2}$$

Notice that Φ_T is a $\mathcal{B}(\mathcal{E}, \mathcal{D}_{m,T^*})$-valued analytic function on $\mathbb{B}^n$. We call Φ_T the *characteristic function* of T corresponding to the characteristic triple $(\mathcal{E}, B, D)$.

We claim that Φ_T is a partially isometric multiplier from $H_n^2(\mathcal{E})$ to $\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})$. To this end, first we proceed to compute $\Phi_T(\boldsymbol{z})\Phi_T(\boldsymbol{w})^*$, $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n$. For simplicity, set

$$x_{\boldsymbol{k}} = \sqrt{\rho_{m-1}(\boldsymbol{k})},$$

and

$$X(\boldsymbol{z}) = \sum_{\boldsymbol{k} \in \mathbb{Z}_+^n} x_{\boldsymbol{k}} D_{\boldsymbol{k}} z^{\boldsymbol{k}}, \qquad Y(\boldsymbol{z}) = D_{m,T^*}(I_{\mathcal{H}} - ZT^*)^{-m} ZB,$$

for all $\boldsymbol{z} \in \mathbb{B}^n$ and $\boldsymbol{k} \in \mathbb{Z}_+^n$. Notice that, if $m = 1$, then $x_{\boldsymbol{k}} = 0$ for all $\boldsymbol{k} \in \mathbb{Z}_+^n \setminus \{0\}$ and $x_{\boldsymbol{0}} = 1$. Thus

$$\Phi_T(\boldsymbol{z})\Phi_T(\boldsymbol{w})^* = X(\boldsymbol{z})X(\boldsymbol{w})^* + X(\boldsymbol{z})Y(\boldsymbol{w})^* + Y(\boldsymbol{z})X(\boldsymbol{w})^* + Y(\boldsymbol{z})Y(\boldsymbol{w})^*,$$

for all $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n$. On the other hand, since $\begin{bmatrix} T^* & B \\ C_{m,T} & D \end{bmatrix}$ is a co-isometry (see Theorem 2.1), we have

$$\begin{bmatrix} T^*T + BB^* & T^*C_{m,T}^* + BD^* \\ C_{m,T}T + DB^* & C_{m,T}C_{m,T}^* + DD^* \end{bmatrix} = \begin{bmatrix} I_{\mathcal{H}^n} & 0 \\ 0 & I_{l^2(\mathbb{Z}_+^n, \mathcal{D}_{m,T^*})} \end{bmatrix}. \tag{3.3}$$

Let $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n$. We note that

$$\begin{aligned} X(\boldsymbol{z})X(\boldsymbol{w})^* &= \sum_{\boldsymbol{k},\boldsymbol{l} \in \mathbb{Z}_+^n} x_{\boldsymbol{k}} x_{\boldsymbol{l}} D_{\boldsymbol{k}} D_{\boldsymbol{l}}^* z^{\boldsymbol{k}} \bar{w}^{\boldsymbol{l}} \\ &= \sum_{\boldsymbol{k} \in \mathbb{Z}_+^n} x_{\boldsymbol{k}}^2 D_{\boldsymbol{k}} D_{\boldsymbol{k}}^* z^{\boldsymbol{k}} \bar{w}^{\boldsymbol{k}} + \sum_{\boldsymbol{k} \neq \boldsymbol{l}} x_{\boldsymbol{k}} x_{\boldsymbol{l}} D_{\boldsymbol{k}} D_{\boldsymbol{l}}^* z^{\boldsymbol{k}} \bar{w}^{\boldsymbol{l}}. \end{aligned}$$

By (3.3), we have $C_{m,T} C_{m,T}^* + DD^* = I_{l^2(\mathbb{Z}_+^n, \mathcal{D}_{m,T^*})}$, which implies

$$x_{\boldsymbol{k}}^2 D_{m,T^*} T^{*\boldsymbol{k}} T^{\boldsymbol{k}} D_{m,T^*} + D_{\boldsymbol{k}} D_{\boldsymbol{k}}^* = I_{\mathcal{D}_{m,T^*}},$$

for all $\boldsymbol{k} \in \mathbb{Z}_+^n$, and

$$x_{\boldsymbol{k}} x_{\boldsymbol{l}} D_{m,T^*} T^{*\boldsymbol{k}} T^{\boldsymbol{l}} D_{m,T^*} + D_{\boldsymbol{k}} D_{\boldsymbol{l}}^* = 0,$$

for all $\boldsymbol{k} \neq \boldsymbol{l}$ in $\mathbb{Z}_+^n$. This implies that

$$\sum_{\boldsymbol{k} \in \mathbb{Z}_+^n} x_{\boldsymbol{k}}^2 D_{\boldsymbol{k}} D_{\boldsymbol{k}}^* z^{\boldsymbol{k}} \bar{w}^{\boldsymbol{k}} = \sum_{\boldsymbol{k} \in \mathbb{Z}_+^n} x_{\boldsymbol{k}}^2 (I_{\mathcal{D}_{m,T^*}} - x_{\boldsymbol{k}}^2 D_{m,T^*} T^{*\boldsymbol{k}} T^{\boldsymbol{k}} D_{m,T^*}) z^{\boldsymbol{k}} \bar{w}^{\boldsymbol{k}},$$

and

$$\sum_{\boldsymbol{k} \neq \boldsymbol{l}} x_{\boldsymbol{k}} x_{\boldsymbol{l}} D_{\boldsymbol{k}} D_{\boldsymbol{l}}^* z^{\boldsymbol{k}} \bar{w}^{\boldsymbol{l}} = - \sum_{\boldsymbol{k} \neq \boldsymbol{l}} x_{\boldsymbol{k}}^2 x_{\boldsymbol{l}}^2 D_{m,T^*} T^{*\boldsymbol{k}} T^{\boldsymbol{l}} D_{m,T^*} z^{\boldsymbol{k}} \bar{w}^{\boldsymbol{l}}.$$

Hence

$$\begin{aligned} X(\boldsymbol{z})X(\boldsymbol{w})^* &= \sum_{\boldsymbol{k} \in \mathbb{Z}_+^n} x_{\boldsymbol{k}}^2 z^{\boldsymbol{k}} \bar{w}^{\boldsymbol{k}} I_{\mathcal{D}_{m,T^*}} - \sum_{\boldsymbol{k},\boldsymbol{l} \in \mathbb{Z}_+^n} x_{\boldsymbol{k}}^2 x_{\boldsymbol{l}}^2 D_{m,T^*} T^{*\boldsymbol{k}} T^{\boldsymbol{l}} D_{m,T^*} z^{\boldsymbol{k}} \bar{w}^{\boldsymbol{l}} \\ &= K_{m-1}(\boldsymbol{z}, \boldsymbol{w}) I_{D_{m,T^*}} - D_{m,T^*} \Big(\sum_{\boldsymbol{k} \in \mathbb{Z}_+^n} x_{\boldsymbol{k}}^2 z^{\boldsymbol{k}} T^{*\boldsymbol{k}} \Big) \Big(\sum_{\boldsymbol{l} \in \mathbb{Z}_+^n} x_{\boldsymbol{l}}^2 \bar{w}^{\boldsymbol{l}} T^{\boldsymbol{l}} \Big) D_{m,T^*} \\ &= K_{m-1}(\boldsymbol{z}, \boldsymbol{w}) I_{D_{m,T^*}} - D_{m,T^*} (I - ZT^*)^{-(m-1)} (I - TW^*)^{-(m-1)} D_{m,T^*}. \end{aligned}$$

Here

$$K_0(\boldsymbol{z}, \boldsymbol{w}) \equiv 1 \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n).$$

Now we compute

$$\begin{aligned} X(\boldsymbol{z})Y(\boldsymbol{w})^* &= \Big(\sum_{\boldsymbol{k} \in \mathbb{Z}_+^n} x_{\boldsymbol{k}} D_{\boldsymbol{k}} z^{\boldsymbol{k}} \Big) \Big(B^* W^* (I - TW^*)^{-m} D_{m,T^*} \Big) \\ &= \sum_{\boldsymbol{k} \in \mathbb{Z}_+^n} x_{\boldsymbol{k}} z^{\boldsymbol{k}} \Big(D_{\boldsymbol{k}} B^* \Big) W^* (I - TW^*)^{-m} D_{m,T^*}. \end{aligned}$$

By (3.3), we have $C_{m,T} T + DB^* = 0$, that is

$$x_{\boldsymbol{k}} D_{m,T^*} T^{*\boldsymbol{k}} T + D_{\boldsymbol{k}} B^* = 0 \qquad (\boldsymbol{k} \in \mathbb{Z}_+^n),$$

and so

$$X(\boldsymbol{z})Y(\boldsymbol{w})^* = -D_{m,T^*} \Big(\sum_{\boldsymbol{k} \in \mathbb{Z}_+^n} x_{\boldsymbol{k}}^2 z^{\boldsymbol{k}} T^{*\boldsymbol{k}} \Big) TW^* (I - TW^*)^{-m} D_{m,T^*},$$

that is

$$X(\boldsymbol{z})Y(\boldsymbol{w})^* = -D_{m,T^*}(I - ZT^*)^{-(m-1)}TW^*(1 - TW^*)^{-m}D_{m,T^*}.$$

By duality

$$Y(\boldsymbol{z})X(\boldsymbol{w})^* = -D_{m,T^*}(I - ZT^*)^{-m}ZT^*(I - TW^*)^{-(m-1)}D_{m,T^*}.$$

Finally, again by (3.3), we have $T^*T + BB^* = I_{\mathcal{H}^n}$, and so

$$\begin{aligned} Y(\boldsymbol{z})Y(\boldsymbol{w})^* &= D_{m,T^*}(I - ZT^*)^{-m}ZBB^*W^*(I - TW^*)^{-m}D_{m,T^*} \\ &= D_{m,T^*}(I - ZT^*)^{-m}Z(I_{\mathcal{H}} - T^*T)W^*(I - TW^*)^{-m}D_{m,T^*}. \end{aligned}$$

Therefore

$$\begin{aligned} \Phi_T(\boldsymbol{z})\Phi_T(\boldsymbol{w})^* = K_{m-1}(\boldsymbol{z},\boldsymbol{w})I_{D_{m,T^*}} &- D_{m,T^*}(I - ZT^*)^{-(m-1)}(I - TW^*)^{-(m-1)}D_{m,T^*} \\ &- D_{m,T^*}(I - ZT^*)^{-(m-1)}TW^*(I - TW^*)^{-m}D_{m,T^*} \\ &- D_{m,T^*}(I - ZT^*)^{-m}ZT^*(I - WT^*)^{-(m-1)}D_{m,T^*} \\ &+ D_{m,T^*}(I - ZT^*)^{-m}Z(I - T^*T)W^*(I - TW^*)^{-m}D_{m,T^*} \\ = K_{m-1}(\boldsymbol{z},\boldsymbol{w})I_{D_{m,T^*}} &- D_{m,T^*}(I - ZT^*)^{-m}M(I - TW^*)^{-m}D_{m,T^*}, \end{aligned}$$

where

$$M = (I - ZT^*)(I - TW^*) + (I - ZT^*)TW^* + ZT^*(I - WT^*) - Z(I - T^*T)W^*.$$

This is now simplified to $M = I - ZW^*$, that is

$$M = (1 - \langle \boldsymbol{z},\boldsymbol{w}\rangle)I,$$

and so

$$\Phi_T(\boldsymbol{z})\Phi_T(\boldsymbol{w})^* = K_{m-1}(\boldsymbol{z},\boldsymbol{w})I_{D_{m,T^*}} - (1 - \langle \boldsymbol{z},\boldsymbol{w}\rangle)D_{m,T^*}(I - ZT^*)^{-m}(I - TW^*)^{-m}D_{m,T^*}.$$

We obtain

$$\frac{1}{(1 - \langle \boldsymbol{z},\boldsymbol{w}\rangle)^m} I_{\mathcal{D}_{m,T^*}} - \frac{\Phi_T(\boldsymbol{z})\Phi_T(\boldsymbol{w})^*}{1 - \langle \boldsymbol{z},\boldsymbol{w}\rangle} = D_{m,T^*}(I - ZT^*)^{-m}(I - TW^*)^{-m}D_{m,T^*}, \tag{3.4}$$

which shows that

$$(\boldsymbol{z},\boldsymbol{w}) \in \mathbb{B}^n \times \mathbb{B}^n \mapsto \frac{1}{(1 - \langle \boldsymbol{z},\boldsymbol{w}\rangle)^m} I_{\mathcal{D}_{m,T^*}} - \frac{\Phi_T(\boldsymbol{z})\Phi_T(\boldsymbol{w})^*}{1 - \langle \boldsymbol{z},\boldsymbol{w}\rangle},$$

is a positive definite kernel. By a well-known fact from reproducing kernel Hilbert space theory (cf. page 2412, [7]), it follows that

$$\Phi_T \in \mathcal{M}(H^2_n(\mathcal{E}), \mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})),$$

and hence

$$M^*_{\Phi_T}\Big(K_m(\cdot,\boldsymbol{w})\eta\Big) = K_1(\cdot,\boldsymbol{w})\Phi_T(\boldsymbol{w})^*\eta \qquad (\boldsymbol{w} \in \mathbb{B}^m, \eta \in \mathcal{D}_{m,T^*}).$$

This shows that

$$(I - M_{\Phi_T}M^*_{\Phi_T})\Big(K_m(\cdot,\boldsymbol{w})\eta\Big)(\boldsymbol{z}) = \Big(K_m(\boldsymbol{z},\boldsymbol{w})I_{\mathcal{D}_{m,T^*}} - K_1(\boldsymbol{z},\boldsymbol{w})\Phi_T(\boldsymbol{z})\Phi_T(\boldsymbol{w})^*\Big)\eta,$$

and hence by (3.4)

$$(I - M_{\Phi_T} M_{\Phi_T}^*)\Big(K_m(\cdot, \boldsymbol{w})\eta\Big)(\boldsymbol{z}) = D_{m,T^*}(I - ZT^*)^{-m}(I - TW^*)^{-m} D_{m,T^*}\eta,$$

for all $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n$ and $\eta \in \mathcal{D}_{m,T^*}$. On the other hand, by the definition of canonical dilations (see (2.1)), $\Pi_m^* : \mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*}) \to \mathcal{H}$ is given by

$$\Pi_m^*\Big(K_m(\cdot, \boldsymbol{w})\eta\Big) = (I_{\mathcal{H}} - TW^*)^{-m} D_{m,T^*}\eta \qquad (\boldsymbol{w} \in \mathbb{B}^m, \eta \in \mathcal{D}_{m,T^*}).$$

This implies that

$$\Pi_m \Pi_m^*\Big(K_m(\cdot, \boldsymbol{w})\eta\Big)(\boldsymbol{z}) = D_{m,T^*}(I_{\mathcal{H}} - ZT^*)^{-m}(I_{\mathcal{H}} - TW^*)^{-m} D_{m,T^*}\eta, \tag{3.5}$$

for all $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n$ and $\eta \in \mathcal{D}_{m,T^*}$, and so

$$\Pi_m \Pi_m^* = I_{\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})} - M_{\Phi_T} M_{\Phi_T}^*.$$

In particular, M_{Φ_T} is a partial isometry and the canonical model invariant subspace corresponding to T (see (2.2)) is given by

$$\mathcal{S}_{m,T} = \Phi_T H_n^2(\mathcal{E}).$$

We have therefore proved the following:

THEOREM 3.1. *Let T be a pure m-hypercontraction on $\mathcal{H}$, and let $(\mathcal{E}, B, D)$ be a characteristic triple of T. Then*

$$\Phi_T \in \mathcal{M}(H_n^2(\mathcal{E}), \mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})),$$

is a partially isometric multiplier and

$$\mathcal{S}_{m,T} = \Phi_T H_n^2(\mathcal{E}),$$

where

$$\Phi_T(z) = \sum_{\boldsymbol{k} \in \mathbb{Z}_+^n} \sqrt{\rho_{m-1}(\boldsymbol{k})} D_{\boldsymbol{k}} z^{\boldsymbol{k}} + D_{m,T^*}(I_{\mathcal{H}} - ZT^*)^{-m} ZB \qquad \big(\boldsymbol{z} \in \mathbb{B}^n\big),$$

is the characteristic function corresponding to $(\mathcal{E}, B, D)$ and $\mathcal{S}_{m,T}$ is the canonical model invariant subspace corresponding to T.

Characteristic triples and functions are more explicit for 1-hypercontractions (or row contractions). This particular case will be discussed in Section 6.

It is worth pointing out, also, that the representing multiplier Φ_T of $\mathcal{S}_{m,T}$ is unique up to a partial isometry constant right factor (cf. [8, Theorem 6.5]): If

$$\mathcal{S}_{m,T} = \tilde{\Phi} H_n^2(\tilde{\mathcal{E}}),$$

for some Hilbert space $\tilde{\mathcal{E}}$ and partially isometric multiplier $\tilde{\Phi} \in \mathcal{M}(H_n^2(\tilde{\mathcal{E}}), \mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*}))$, then there exists a partial isometry $V \in \mathcal{B}(\tilde{\mathcal{E}}, \mathcal{E})$ such that

$$\tilde{\Phi}(\boldsymbol{z}) = \Phi_T(\boldsymbol{z}) V \qquad (\boldsymbol{z} \in \mathbb{B}^n).$$

We now proceed to prove complete unitary invariance of characteristic triples of pure m-hypercontractions. Recall that two commuting tuples $T = (T_1, \cdots, T_n)$ on $\mathcal{H}$ and $\tilde{T} =$

$(\tilde{T}_1, \dots, \tilde{T}_n)$ on $\tilde{\mathcal{H}}$ are said to be *unitarily equivalent* if there exists a unitary $U \in \mathcal{B}(\mathcal{H}, \tilde{\mathcal{H}})$ such that $UT_i = \tilde{T}_i U$ for all $i = 1, \dots, n$.

Let T and $\tilde{T}$ be pure m-hypercontractions on $\mathcal{H}$ and $\tilde{\mathcal{H}}$, respectively. Let Φ_T and $\Phi_{\tilde{T}}$ be characteristic functions corresponding to characteristic triples $(\mathcal{E}, B, D)$ and $(\tilde{\mathcal{E}}, \tilde{B}, \tilde{D})$ of T and $\tilde{T}$, respectively. The characteristic functions Φ_T and $\Phi_{\tilde{T}}$ are said to *coincide* if

$$\Phi_{\tilde{T}}(\boldsymbol{z}) = \tau_* \Phi_T(\boldsymbol{z}) \tau \qquad (\boldsymbol{z} \in \mathbb{B}^n),$$

for some unitary operators $\tau : \tilde{\mathcal{E}} \to \mathcal{E}$ and $\tau_* : \mathcal{D}_{m,T^*} \to \mathcal{D}_{m,\tilde{T}^*}$. Characteristic triples of pure m-hypercontractions are complete unitary invariants:

Theorem 3.2. *Let T and $\tilde{T}$ be pure m-hypercontractions on $\mathcal{H}$ and $\tilde{\mathcal{H}}$, respectively. Then T and $\tilde{T}$ are unitarily equivalent if and only if characteristic functions of T and $\tilde{T}$ coincide.*

Proof. Let Φ_T and $\Phi_{\tilde{T}}$ be characteristic functions corresponding to characteristic triples $(\mathcal{E}, B, D)$ and $(\tilde{\mathcal{E}}, \tilde{B}, \tilde{D})$ of T and $\tilde{T}$, respectively. Then

$$U = \begin{bmatrix} X_T & Y_T \end{bmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{E}, \mathcal{H}^n \oplus l^2(\mathbb{Z}_+^n, \mathcal{D}_{m,T^*})),$$

and

$$\tilde{U} = \begin{bmatrix} X_{\tilde{T}} & Y_{\tilde{T}} \end{bmatrix} \in \mathcal{B}(\tilde{\mathcal{H}} \oplus \tilde{\mathcal{E}}, \tilde{\mathcal{H}}^n \oplus l^2(\mathbb{Z}_+^n, \mathcal{D}_{m,\tilde{T}^*}),$$

are unitaries corresponding to characteristic triples $(\mathcal{E}, B, D)$ and $(\tilde{\mathcal{E}}, \tilde{B}, \tilde{D})$, respectively, as in Theorem 2.1.

To prove the forward implication, let $W : \mathcal{H} \to \tilde{\mathcal{H}}$ be a unitary such that $WT_i = \tilde{T}_i W$, $i = 1, \dots, n$. Then $WD_{m,T^*} = D_{m,\tilde{T}^*} W$, and so

$$C_{m,\tilde{T}} W = (I \otimes W|_{\mathcal{D}_{m,T^*}}) C_{m,T}.$$

Also we have unitaries

$$W_n := W \oplus \cdots \oplus W : \mathcal{H}^n \to \tilde{\mathcal{H}}^n,$$

and

$$\hat{W} := \begin{bmatrix} W_n & 0 \\ 0 & I \otimes W|_{\mathcal{D}_{m,T^*}} \end{bmatrix} : \mathcal{H}^n \oplus l^2(\mathbb{Z}_+^n, \mathcal{D}_{m,T^*}) \to \tilde{\mathcal{H}}^n \oplus l^2(\mathbb{Z}_+^n, \mathcal{D}_{m,\tilde{T}^*}),$$

which gives

$$W_n T^* = \tilde{T}^* W \quad \text{and} \quad \hat{W} X_T = X_{\tilde{T}} W.$$

Hence

$$\begin{bmatrix} X_{\tilde{T}} & \hat{W} Y_T \end{bmatrix} = \hat{W} \begin{bmatrix} X_T & Y_T \end{bmatrix} \begin{bmatrix} W^* & 0 \\ 0 & I_{\mathcal{E}} \end{bmatrix}.$$

In particular

$$\begin{bmatrix} X_{\tilde{T}} & \hat{W} Y_T \end{bmatrix} : \tilde{\mathcal{H}} \oplus \mathcal{E} \to \tilde{\mathcal{H}}^n \oplus l^2(\mathbb{Z}_+^n, \mathcal{D}_{m,\tilde{T}^*}),$$

is a unitary and

$$\hat{W} Y_T = \begin{bmatrix} W_n B \\ (I \otimes W) D \end{bmatrix} \in \mathcal{B}(\mathcal{E}, \tilde{\mathcal{H}}^n \oplus l^2(\mathbb{Z}_+^n, \mathcal{D}_{m,\tilde{T}^*})),$$

is an isometry. Thus, $(\mathcal{E}, W_n B, (I \otimes W)D)$ is a characteristic triple of $\tilde{T}$ and hence, by Theorem 2.2, there exists a unitary $V : \mathcal{E} \to \tilde{\mathcal{E}}$ such that $\hat{W} Y_T = Y_{\tilde{T}} V$. This shows that

$$Y_{\tilde{T}} = \hat{W} Y_T V^*,$$

that is

$$Y_{\tilde{T}} = \begin{bmatrix} W_n B V^* \\ (I \otimes W) D V^* \end{bmatrix},$$

and so

$$\tilde{U} = \begin{bmatrix} \tilde{T}^* & W_n B V^* \\ C_{m,\tilde{T}} & (I \otimes W) D V^* \end{bmatrix}.$$

A routine computation then shows that

$$\Phi_{\tilde{T}}(\boldsymbol{z}) = W \Phi_T(\boldsymbol{z}) V^* \qquad (\boldsymbol{z} \in \mathbb{B}^n).$$

In order to prove sufficiency, we let $\Phi_{\tilde{T}}(\boldsymbol{z}) = \tau_* \Phi_T(\boldsymbol{z}) \tau^*$ for all $\boldsymbol{z} \in \mathbb{B}^n$ for some unitaries $\tau \in \mathcal{B}(\mathcal{E}, \tilde{\mathcal{E}})$ and $\tau_* \in \mathcal{B}(\mathcal{D}_{m,T^*}, \mathcal{D}_{m,\tilde{T}^*})$. Then

$$M_{\Phi_{\tilde{T}}} = (I_{\mathbb{H}_m} \otimes \tau_*) M_{\Phi_T} (I_{H^2_n} \otimes \tau^*),$$

and so

$$(I_{\mathbb{H}_m} \otimes \tau_*^*)(I_{\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,\tilde{T}^*})} - M_{\Phi_{\tilde{T}}} M_{\Phi_{\tilde{T}}}^*) = (I_{\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})} - M_{\Phi_T} M_{\Phi_T}^*)(I_{\mathbb{H}_m} \otimes \tau_*^*),$$

that is

$$(I_{\mathbb{H}_m} \otimes \tau_*^*) P_{\mathcal{Q}_{\tilde{T}}} = P_{\mathcal{Q}_T} (I_{\mathbb{H}_m} \otimes \tau_*^*).$$

It follows that

$$(I_{\mathbb{H}_m} \otimes \tau_*^*) \mathcal{Q}_{\tilde{T}} = \mathcal{Q}_T.$$

Moreover

$$\begin{aligned} (I_{\mathbb{H}_m} \otimes \tau_*^*)\Big(P_{\mathcal{Q}_{\tilde{T}}} M_{z_i} P_{\mathcal{Q}_{\tilde{T}}}\Big) &= (I_{\mathbb{H}_m} \otimes \tau_*^*) P_{\mathcal{Q}_{\tilde{T}}} M_{z_i} P_{\mathcal{Q}_{\tilde{T}}} \\ &= P_{\mathcal{Q}_T} (I_{\mathbb{H}_m} \otimes \tau_*^*) M_{z_i} P_{\mathcal{Q}_{\tilde{T}}} \\ &= P_{\mathcal{Q}_T} M_{z_i} (I_{\mathbb{H}_m} \otimes \tau_*^*) P_{\mathcal{Q}_{\tilde{T}}}, \end{aligned}$$

that is

$$(I_{\mathbb{H}_m} \otimes \tau_*^*)\Big(P_{\mathcal{Q}_{\tilde{T}}} M_{z_i} P_{\mathcal{Q}_{\tilde{T}}}\Big) = \Big(P_{\mathcal{Q}_T} M_{z_i} P_{\mathcal{Q}_T}\Big)(I_{\mathbb{H}_m} \otimes \tau_*^*),$$

for all $i = 1, \ldots, n$. Combining with the previous equality, we conclude that

$$P_{\mathcal{Q}_T} M_z|_{\mathcal{Q}_T} \cong P_{\mathcal{Q}_{\tilde{T}}} M_z|_{\mathcal{Q}_{\tilde{T}}},$$

that is, $T \cong \tilde{T}$. ∎

We now proceed to study joint invariant subspaces of pure m-hypercontractions. Following Sz.-Nagy-Foias factorizations of characteristic functions, we relate joint invariant subspaces of pure m-hypercontractions with operator-valued factors of characteristic functions corresponding to characteristic triples. We make good use of the following fact (see Lemma 2, [3]):

Lemma 3.3. *Let $\mathcal{E}$, $\mathcal{E}_*$ and $\mathcal{F}$ be Hilbert spaces, and let Φ and Ψ be $\mathcal{B}(\mathcal{E}, \mathcal{E}_*)$ and $\mathcal{B}(\mathcal{F}, \mathcal{E}_*)$ valued analytic functions, respectively, on $\mathbb{B}^n$. Then the following are equivalent:*

(i) $(\boldsymbol{z},\boldsymbol{w}) \mapsto \frac{\Psi(\boldsymbol{z})\Psi(\boldsymbol{w})^*-\Phi(\boldsymbol{z})\Phi(\boldsymbol{w})^*}{1-\langle \boldsymbol{z},\boldsymbol{w}\rangle}$ *is a positive-definite kernel on* $\mathbb{B}^n$*.*

(ii) *There exists a contractive multiplier* $\Theta \in \mathcal{M}(H^2_n(\mathcal{E}), H^2_n(\mathcal{F}))$ *such that*

$$\Phi(z) = \Psi(\boldsymbol{z})\Theta(\boldsymbol{z}) \qquad (\boldsymbol{z} \in \mathbb{B}^n).$$

We are now ready for a factorization theorem for joint invariant subspaces of pure m-hypercontractions.

THEOREM 3.4. *Let T be a pure m-hypercontraction on $\mathcal{H}$, and let $(\mathcal{E}, B, D)$ be a characteristic triple of T. If Φ_T is the characteristic function corresponding to $(\mathcal{E}, B, D)$, then T has a closed joint invariant subspace if and only if there exist a Hilbert space $\mathcal{F}$, a contractive multiplier $\Phi_1 \in \mathcal{M}(H^2_n(\mathcal{E}), H^2_n(\mathcal{F}))$, and a partially isometric multiplier $\Phi_2 \in \mathcal{M}(H^2_n(\mathcal{F}), \mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*}))$ such that*

$$\Phi_T(\boldsymbol{z}) = \Phi_2(\boldsymbol{z})\Phi_1(\boldsymbol{z}) \qquad (\boldsymbol{z} \in \mathbb{B}^n).$$

Moreover, the joint-invariant subspace is non-trivial if and only if ranM_{Φ_2} is neither equal to ranM_{Φ_T} nor to $\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})$.

Proof. Let $\mathcal{H}_1$ be a closed joint T-invariant subspace of $\mathcal{H}$, and let $\mathcal{H}_2 = \mathcal{H} \ominus \mathcal{H}_1$. Then

$$\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*}) \ominus \Pi_m \mathcal{H}_2,$$

is a joint M_z-invariant subspace of $\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})$. By a Beurling-Lax-Halmos type theorem for weighted Bergman spaces (see Theorem 4.4, [21]), there exist a Hilbert space $\mathcal{F}$ and a partially isometric multiplier $\Phi_2 \in \mathcal{M}(H^2_n(\mathcal{F}), \mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*}))$ such that

$$\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*}) \ominus \Pi_m \mathcal{H}_2 = \Phi_2 H^2_n(\mathcal{F}). \tag{3.6}$$

Since $\mathcal{Q}_T = \Pi_m \mathcal{H}$ and $\Pi_m \mathcal{H} = \mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*}) \ominus \Phi_T H^2_n(\mathcal{E})$, we conclude that

$$\begin{aligned}
\Pi_m \mathcal{H}_1 &= \Pi_m \mathcal{H} \ominus \Pi_m \mathcal{H}_2 \\
&= \Big(\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*}) \ominus \Phi_T H^2_n(\mathcal{E})\Big) \ominus \Big(\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*}) \ominus \Phi_2 H^2_n(\mathcal{F})\Big) \\
&= \Phi_2 H^2_n(\mathcal{F}) \ominus \Phi_T H^2_n(\mathcal{E}), \qquad (3.7)
\end{aligned}$$

and hence

$$(\boldsymbol{z},\boldsymbol{w}) \in \mathbb{B}^n \times \mathbb{B}^n \mapsto \frac{\Phi_2(\boldsymbol{z})\Phi_2(\boldsymbol{w})^* - \Phi_T(\boldsymbol{z})\Phi_T(\boldsymbol{w})^*}{1-\langle \boldsymbol{z},\boldsymbol{w}\rangle},$$

is a kernel of the reproducing kernel Hilbert space $\Pi_m \mathcal{H}_1$. By Lemma 3.3, there is a contractive multiplier $\Phi_1 \in \mathcal{M}(H^2_n(\mathcal{E}), H^2_n(\mathcal{F}))$ such that $\Phi_T(\boldsymbol{z}) = \Phi_2(\boldsymbol{z})\Phi_1(\boldsymbol{z})$ for all $\boldsymbol{z} \in \mathbb{B}^n$.
To prove the converse, let $\mathcal{F}$ be a Hilbert space, $\Phi_1 \in \mathcal{M}(H^2_n(\mathcal{E}), H^2_n(\mathcal{F}))$ be a contractive multiplier, $\Phi_2 \in \mathcal{M}(H^2_n(\mathcal{F}), \mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*}))$ be a partially isometric multiplier, and let $\Phi_T(\boldsymbol{z}) = \Phi_2(\boldsymbol{z})\Phi_1(\boldsymbol{z})$ for all $\boldsymbol{z} \in \mathbb{B}^n$. We have

$$\text{ran} M_{\Phi_T} \subseteq \text{ran} M_{\Phi_2},$$

and hence

$$\mathcal{Q} \subseteq \mathcal{Q}_T,$$

where

$$\mathcal{Q} = \Big(\text{ran} M_{\Phi_2}\Big)^{\perp},$$

is a joint M_z^*-invariant subspace of $\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})$. It now follows that $\mathcal{H}_1 = \mathcal{H} \ominus \Pi_m^* \mathcal{Q}$ is a joint T-invariant subspace of $\mathcal{H}$.
For the last part, note that, by (3.6), the invariant subspace $\mathcal{H}_1$ of T is the full space if and only if $\text{ran} M_{\Phi_2} = \mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})$. On the other hand, by (3.7), $\mathcal{H}_1 = 0$ if and only if $\text{ran} M_{\Phi_2} = \text{ran} M_{\Phi_T}$. This completes the proof of the theorem. ■

4. Universal multipliers and wandering subspaces

In [7], Ball and Bolotnikov proved the following: Given a vector-valued weighted shift space $H^2(\beta, \mathcal{E}_*)$ (see the definition below), there exists a *universal multiplier* ψ_β (depending only on β and $\mathcal{E}_*$) such that any contractive multiplier θ from a vector-valued Hardy space $H^2_{\mathcal{E}}(\mathbb{D})$ to $H^2(\beta, \mathcal{E}_*)$ factors through ψ_β, that is

$$\theta(z) = \psi_\beta(z)\tilde{\theta}(z) \qquad (z \in \mathbb{D}),$$

for some Schur multiplier $\tilde{\theta} \in \mathcal{M}(H^2_{\mathcal{E}}(\mathbb{D}), H^2_{l^2(\mathcal{E}_*)}(\mathbb{D}))$ (see [7, Theorem 2.1] for more details).

In this section, we generalize the above to several variables multipliers. We also define "inner functions" and examine the uniqueness of universal factorizations in several variables. First, we fix some notation and terminology.

A strictly decreasing sequence of positive numbers $\beta = \{\beta_j\}_{j=0}^\infty$ is said to be a *weight sequence*, if $\beta_0 = 1$ and

$$\liminf \beta_j^{\frac{1}{j}} \geq 1. \tag{4.1}$$

For a Hilbert space $\mathcal{E}$ and a weight sequence β, we let $\mathbb{H}^2_n(\beta, \mathcal{E})$ denote the Hilbert space of all $\mathcal{E}$-valued analytic functions $f = \sum\limits_{\boldsymbol{k} \in \mathbb{Z}_+^n} a_{\boldsymbol{k}} z^{\boldsymbol{k}}$, $a_{\boldsymbol{k}} \in \mathcal{E}$ for all $\boldsymbol{k} \in \mathbb{Z}_+^n$, on $\mathbb{B}^n$ such that

$$\|f\|^2_{\mathbb{H}^2_n(\beta,\mathcal{E})} := \sum_{j=0}^\infty \beta_j \sum_{|\boldsymbol{k}|=j} \frac{1}{\rho_1(\boldsymbol{k})} \|a_{\boldsymbol{k}}\|^2_{\mathcal{E}} = \sum_{\boldsymbol{k} \in \mathbb{Z}_+^n} \frac{\beta_{|\boldsymbol{k}|}}{\rho_1(\boldsymbol{k})} \|a_{\boldsymbol{k}}\|^2_{\mathcal{E}} < \infty,$$

that is

$$\mathbb{H}^2_n(\beta, \mathcal{E}) = \Big\{ f \in \mathcal{O}(\mathbb{B}^n, \mathcal{E}) : \|f\|_{\mathbb{H}^2_n(\beta,\mathcal{E})} < \infty \Big\}.$$

Then $\mathbb{H}^2_n(\beta, \mathcal{E})$ is an $\mathcal{E}$-valued reproducing kernel Hilbert space corresponding to the kernel

$$K_\beta(\boldsymbol{z}, \boldsymbol{w}) = \sum_{j=0}^\infty \frac{1}{\beta_j} \langle \boldsymbol{z}, \boldsymbol{w} \rangle^j I_{\mathcal{E}} \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n). \tag{4.2}$$

In particular, for $\beta_j = \frac{j!(n-1)!}{(n+j-1)!}$ and $\beta_j = \frac{j!n!}{(n+j)!}$, $j \in \mathbb{Z}_+$, $\mathbb{H}^2_n(\beta, \mathcal{E})$ represents the $\mathcal{E}$-valued Hardy space and the Bergman space over $\mathbb{B}^n$, respectively.

We now proceed to construct the universal multiplier corresponding to the weight sequence β and the Hilbert space $\mathcal{E}$. Let

$$\gamma_0 = 1 \quad \text{ and } \gamma_j = \Big(\frac{1}{\beta_j} - \frac{1}{\beta_{j-1}} \Big)^{-1} \qquad (j \geq 1).$$

Then $\gamma = \{\gamma_j\}_{j\in\mathbb{Z}_+}$ is also a weight sequence and hence

$$K_\gamma(\boldsymbol{z},\boldsymbol{w}) = \sum_{j=0}^{\infty} \frac{1}{\gamma_j}\langle \boldsymbol{z},\boldsymbol{w}\rangle^j \qquad (\boldsymbol{z},\boldsymbol{w}\in\mathbb{B}^n),$$

is a positive-definite kernel on $\mathbb{B}^n$. Define $\Psi_{\beta,\mathcal{E}} : \mathbb{B}^n \to \mathcal{B}(l^2(\mathbb{Z}_+^n,\mathcal{E}),\mathcal{E})$ by

$$\Psi_{\beta,\mathcal{E}}(\boldsymbol{z})(\{a_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}_+^n}) = \sum_{\boldsymbol{k}\in\mathbb{Z}_+^n}\Big(\sqrt{\frac{\rho_1(\boldsymbol{k})}{\gamma_{|\boldsymbol{k}|}}}a_{\boldsymbol{k}}\Big)z^{\boldsymbol{k}},$$

for all $\boldsymbol{z}\in\mathbb{B}^n$ and $\{a_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}_+^n}\in l^2(\mathbb{Z}_+^n,\mathcal{E})$. We must first show that $\Psi_{\beta,\mathcal{E}}$ is well-defined. For each $\boldsymbol{z}\in\mathbb{B}^n$ and $\{a_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}_+^n}\in l^2(\mathbb{Z}_+^n,\mathcal{E})$, we have

$$\begin{aligned}
\Big\|\sum_{\boldsymbol{k}\in\mathbb{Z}_+^n}\Big(\sqrt{\frac{\rho_1(\boldsymbol{k})}{\gamma_{|\boldsymbol{k}|}}}a_{\boldsymbol{k}}\Big)z^{\boldsymbol{k}}\Big\|_{\mathcal{E}} &\le \sum_{\boldsymbol{k}\in\mathbb{Z}_+^n}\sqrt{\frac{\rho_1(\boldsymbol{k})}{\gamma_{|\boldsymbol{k}|}}}|z^{\boldsymbol{k}}|\|a_{\boldsymbol{k}}\|_{\mathcal{E}}\\
&\le \Big(\sum_{\boldsymbol{k}\in\mathbb{Z}_+^n}\frac{\rho_1(\boldsymbol{k})}{\gamma_{|\boldsymbol{k}|}}|z|^{2\boldsymbol{k}}\Big)^{\frac{1}{2}}\Big(\sum_{\boldsymbol{k}\in\mathbb{Z}_+^n}\|a_{\boldsymbol{k}}\|_{\mathcal{E}}^2\Big)^{\frac{1}{2}}\\
&= \Big(\sum_{j=0}^{\infty}\frac{1}{\gamma_j}\langle\boldsymbol{z},\boldsymbol{z}\rangle^j\Big)^{\frac{1}{2}}\|\{a_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}_+^n}\|_{l^2(\mathbb{Z}_+^n,\mathcal{E})}\\
&= K_\gamma(\boldsymbol{z},\boldsymbol{z})^{\frac{1}{2}}\|\{a_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}_+^n}\|_{l^2(\mathbb{Z}_+^n,\mathcal{E})},
\end{aligned}$$

that is

$$\|\Psi_{\beta,\mathcal{E}}(\boldsymbol{z})(\{a_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}_+^n})\| \le K_\gamma(\boldsymbol{z},\boldsymbol{z})^{\frac{1}{2}}\|\{a_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}_+^n}\|_{l^2(\mathbb{Z}_+^n,\mathcal{E})}.$$

It is again convenient to represent $\Psi_\beta(\boldsymbol{z})$, $\boldsymbol{z}\in\mathbb{B}^n$, as the row operator

$$\Psi_{\beta,\mathcal{E}}(\boldsymbol{z}) = \Big[\cdots\sqrt{\frac{\rho_1(\boldsymbol{k})}{\gamma_{|\boldsymbol{k}|}}}z^{\boldsymbol{k}}I_{\mathcal{E}}\cdots\Big]_{\boldsymbol{k}\in\mathbb{Z}_+^n}.$$

Now we prove that:

Lemma 4.1. *$\Psi_{\beta,\mathcal{E}}\in\mathcal{M}(H_n^2(l^2(\mathbb{Z}_+^n,\mathcal{E})),\mathbb{H}_n^2(\beta,\mathcal{E}))$ and*

$$M_{\Psi_{\beta,\mathcal{E}}} : H_n^2\big(l^2(\mathbb{Z}_+^n,\mathcal{E})\big)\to\mathbb{H}_n^2(\beta,\mathcal{E}),$$

is a co-isometry.

Proof: For $\boldsymbol{z}$ and $\boldsymbol{w}$ in $\mathbb{B}^n$, we have

$$\begin{aligned}(1-\langle \boldsymbol{z},\boldsymbol{w}\rangle)K_\beta(\boldsymbol{z},\boldsymbol{w}) &= \sum_{j=0}^\infty \frac{1}{\beta_j}\langle \boldsymbol{z},\boldsymbol{w}\rangle^j - \sum_{j=0}^\infty \frac{1}{\beta_j}\langle \boldsymbol{z},\boldsymbol{w}\rangle^{j+1} \\ &= \frac{1}{\beta_0} + \sum_{j=0}^\infty \Big(\frac{1}{\beta_{j+1}}\langle \boldsymbol{z},\boldsymbol{w}\rangle^{j+1} - \frac{1}{\beta_j}\langle \boldsymbol{z},\boldsymbol{w}\rangle^{j+1}\Big) \\ &= \frac{1}{\beta_0} + \sum_{j=0}^\infty \Big(\frac{1}{\beta_{j+1}} - \frac{1}{\beta_j}\Big)\langle \boldsymbol{z},\boldsymbol{w}\rangle^{j+1} \\ &= \sum_{j=0}^\infty \frac{1}{\gamma_j}\langle \boldsymbol{z},\boldsymbol{w}\rangle^j,\end{aligned}$$

that is

$$(1-\langle \boldsymbol{z},\boldsymbol{w}\rangle)K_\beta(\boldsymbol{z},\boldsymbol{w}) = K_\gamma(\boldsymbol{z},\boldsymbol{w}).$$

Hence from the matrix representation of $\Psi_{\beta,\mathcal{E}}$ it follows that

$$\begin{aligned}\Psi_{\beta,\mathcal{E}}(\boldsymbol{z})\Psi_{\beta,\mathcal{E}}(\boldsymbol{w})^* &= \sum_{j=0}^\infty \frac{1}{\gamma_j}\sum_{|\boldsymbol{k}|=j}\rho_{\boldsymbol{k}} z^{\boldsymbol{k}}\bar{w}^{\boldsymbol{k}} I_{\mathcal{E}} \\ &= \sum_{j=0}^\infty \frac{1}{\gamma_j}\langle \boldsymbol{z},\boldsymbol{w}\rangle^j I_{\mathcal{E}} \\ &= K_\gamma(\boldsymbol{z},\boldsymbol{w})I_{\mathcal{E}} \\ &= (1-\langle \boldsymbol{z},\boldsymbol{w}\rangle)K_\beta(\boldsymbol{z},\boldsymbol{w})I_{\mathcal{E}},\end{aligned}$$

which implies

$$K_\beta(\boldsymbol{z},\boldsymbol{w})I_{\mathcal{E}} - \frac{\Psi_\beta(\boldsymbol{z})\Psi_\beta(\boldsymbol{w})^*}{1-\langle \boldsymbol{z},\boldsymbol{w}\rangle} = 0, \tag{4.3}$$

and so $\Psi_{\beta,\mathcal{E}} \in \mathcal{M}(H^2_n\big(l^2(\mathbb{Z}^n_+,\mathcal{E})\big), \mathbb{H}^2_n(\beta,\mathcal{E}))$. The remaining part of the lemma follows from (4.3) and the fact that $\{K_\beta(\cdot,\boldsymbol{w})\eta : \boldsymbol{w}\in\mathbb{B}^n, \eta\in\mathcal{E}\}$ is a total set in $H^2_n(\beta,\mathcal{E})$. ∎

Given Hilbert spaces $\mathcal{E}$ and $\mathcal{E}_*$, we use $\mathcal{SM}(H^2_n(\mathcal{E}),\mathbb{H}^2_n(\beta,\mathcal{E}_*))$ to denote the set of all contractive multipliers, that is

$$\mathcal{SM}(H^2_n(\mathcal{E}),\mathbb{H}^2_n(\beta,\mathcal{E}_*)) = \{\Phi\in\mathcal{M}(H^2_n(\mathcal{E}),\mathbb{H}^2_n(\beta,\mathcal{E}_*)) : \|M_\Phi\|\le 1\}.$$

Now we are ready to prove the main theorem of this section.

Theorem 4.2. *Let $\mathcal{E}$ and $\mathcal{E}_*$ be Hilbert spaces, β be a weight sequence, and let $\Theta : \mathbb{B}^n \to \mathcal{B}(\mathcal{E},\mathcal{E}_*)$ be an analytic function. Then $\Theta\in\mathcal{SM}(H^2_n(\mathcal{E}),\mathbb{H}^2_n(\beta,\mathcal{E}_*))$ if and only if there exists a multiplier $\tilde{\Theta}\in\mathcal{SM}(H^2_n(\mathcal{E}),H^2_n(l^2(\mathbb{Z}^n_+,\mathcal{E}_*))$ such that*

$$\Theta(\boldsymbol{z}) = \Psi_{\beta,\mathcal{E}_*}(\boldsymbol{z})\tilde{\Theta}(\boldsymbol{z}) \qquad (\boldsymbol{z}\in\mathbb{B}^n).$$

Proof: Let $\tilde{\Theta} \in \mathcal{SM}(H^2_n(\mathcal{E}), H^2_n(l^2(\mathbb{Z}^n_+, \mathcal{E}_*))$, and let $\Theta(\boldsymbol{z}) = \Psi_{\beta,\mathcal{E}_*}(\boldsymbol{z})\tilde{\Theta}(\boldsymbol{z})$ for all $\boldsymbol{z} \in \mathbb{B}^n$. Then

$$\begin{aligned} K_\beta(\boldsymbol{z},\boldsymbol{w})I_{\mathcal{E}_*} - \frac{\Theta(\boldsymbol{z})\Theta(\boldsymbol{w})^*}{1-\langle \boldsymbol{z},\boldsymbol{w}\rangle} &= K_\beta(\boldsymbol{z},\boldsymbol{w})I_{\mathcal{E}_*} - \frac{\Psi_{\beta,\mathcal{E}_*}(\boldsymbol{z})\tilde{\Theta}(\boldsymbol{z})\tilde{\Theta}(\boldsymbol{w})^*\Psi_{\beta,\mathcal{E}_*}(\boldsymbol{w})^*}{1-\langle \boldsymbol{z},\boldsymbol{w}\rangle} \\ &= \frac{\Psi_{\beta,\mathcal{E}_*}(\boldsymbol{z})\Psi_{\beta,\mathcal{E}_*}(\boldsymbol{w})^*}{1-\langle \boldsymbol{z},\boldsymbol{w}\rangle} - \frac{\Psi_{\beta,\mathcal{E}_*}(\boldsymbol{z})\tilde{\Theta}(\boldsymbol{z})\tilde{\Theta}(\boldsymbol{w})^*\Psi_{\beta,\mathcal{E}_*}(\boldsymbol{w})^*}{1-\langle \boldsymbol{z},\boldsymbol{w}\rangle} \\ &= \Psi_{\beta,\mathcal{E}_*}(\boldsymbol{z})\Big[\frac{I_{l^2(\mathbb{Z}^n_+,\mathcal{E}_*)} - \tilde{\Theta}(\boldsymbol{z})\tilde{\Theta}(\boldsymbol{w})^*}{1-\langle \boldsymbol{z},\boldsymbol{w}\rangle}\Big]\Psi_{\beta,\mathcal{E}_*}(\boldsymbol{w})^*, \end{aligned}$$

for all $\boldsymbol{z},\boldsymbol{w} \in \mathbb{B}^n$, where the last but one equality follows from (4.3). Since $\tilde{\Theta}$ is a contractive multiplier, it follows that

$$(\boldsymbol{z},\boldsymbol{w}) \mapsto K_\beta(\boldsymbol{z},\boldsymbol{w})I_{\mathcal{E}_*} - \frac{\Theta(\boldsymbol{z})\Theta(\boldsymbol{w})^*}{1-\langle \boldsymbol{z},\boldsymbol{w}\rangle},$$

is a positive definite kernel on $\mathbb{B}^n$, and so $\Theta \in \mathcal{SM}(H^2_n(\mathcal{E}), \mathbb{H}^2_n(\beta,\mathcal{E}_*))$. To prove the converse we first note that $M_\Theta : H^2_n(\mathcal{E}) \to \mathbb{H}^2_n(\beta,\mathcal{E}_*)$ is a contraction. Again, by (4.3), we have

$$\begin{aligned} K_\beta(\boldsymbol{z},\boldsymbol{w})I_{\mathcal{E}_*} - \frac{\Theta(\boldsymbol{z})\Theta(\boldsymbol{w})^*}{1-\langle \boldsymbol{z},\boldsymbol{w}\rangle} &= \frac{\Psi_{\beta,\mathcal{E}_*}(\boldsymbol{z})\Psi_{\beta,\mathcal{E}_*}(\boldsymbol{w})^*}{1-\langle \boldsymbol{z},\boldsymbol{w}\rangle} - \frac{\Theta(\boldsymbol{z})\Theta(\boldsymbol{w})^*}{1-\langle \boldsymbol{z},\boldsymbol{w}\rangle} \\ &= \frac{\Psi_{\beta,\mathcal{E}_*}(\boldsymbol{z})\Psi_{\beta,\mathcal{E}_*}(\boldsymbol{w})^* - \Theta(\boldsymbol{z})\Theta(\boldsymbol{w})^*}{1-\langle \boldsymbol{z},\boldsymbol{w}\rangle}, \end{aligned}$$

for all $\boldsymbol{z},\boldsymbol{w} \in \mathbb{B}^n$. Hence

$$(\boldsymbol{z},\boldsymbol{w}) \mapsto \frac{\Psi_{\beta,\mathcal{E}_*}(\boldsymbol{z})\Psi_{\beta,\mathcal{E}_*}(\boldsymbol{w})^* - \Theta(\boldsymbol{z})\Theta(\boldsymbol{w})^*}{1-\langle \boldsymbol{z},\boldsymbol{w}\rangle} \in \mathcal{B}(\mathcal{E}_*),$$

is a positive-definite kernel on $\mathbb{B}^n$. The proof now follows from Lemma 3.3. ∎

The above theorem implies that the following diagram is commutative:

$$\begin{tikzcd} & H^2_n(l^2(\mathbb{Z}^n_+,\mathcal{E}_*)) \arrow[d, "M_{\Psi_{\beta,\mathcal{E}_*}}"] \\ H^2_n(\mathcal{E}) \arrow[ur, "M_{\tilde{\Theta}}"] \arrow[r, "M_\Theta"'] & \mathbb{H}^2_n(\beta,\mathcal{E}_*) \end{tikzcd}$$

We now turn to "inner functions" in $\mathcal{M}(H^2_n(\mathcal{E}), \mathbb{H}^2_n(\beta,\mathcal{E}_*))$. The concept of inner functions in the setting of Bergman space (knows as the Bergman inner functions) is due to Hedenmalm [12] (see also Olofsson [17] and Eschmeier [10] for weighted Bergman spaces in one and several variables, respectively). The notion of inner functions (or K-inner functions) in several variables was introduced in [8].

A contractive multiplier $\Theta \in \mathcal{M}(H^2_n(\mathcal{E}), \mathbb{H}^2_n(\beta, \mathcal{E}_*))$ is said to be K_β-*inner* if

$$\|\Theta h\|_{\mathbb{H}^2_n(\beta,\mathcal{E}_*)} = \|h\|_{\mathcal{E}},$$

for all $h \in \mathcal{E}$ (that is, $M_\Theta|_{\mathcal{E}}$ is an isometry), and

$$\Theta\mathcal{E} \perp z^{\boldsymbol{k}}\Theta\mathcal{E} \qquad (\boldsymbol{k} \in \mathbb{Z}^n_+).$$

In the case when $\Theta \in \mathcal{M}(H^2_n(\mathcal{E}), H^2_n(\mathcal{E}_*))$ we simply say Θ is a K-inner multiplier.

In connection with this notice also that for a Hilbert space $\mathcal{E}$

$$M_{\Psi_{\beta,\mathcal{E}}}|_{l^2(\mathbb{Z}^n_+,\mathcal{E})} : l^2(\mathbb{Z}^n_+, \mathcal{E}) \to \mathbb{H}^2_n(\gamma, \mathcal{E}),$$

is an isometry. Indeed, for each $\{a_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^n_+} \in l^2(\mathbb{Z}^n_+, \mathcal{E})$ and $\boldsymbol{z} \in \mathbb{B}^n$, we have

$$(M_{\Psi_{\beta,\mathcal{E}}}(\{a_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^n_+}))(\boldsymbol{z}) = \Psi_{\beta,\mathcal{E}}(\boldsymbol{z})(\{a_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^n_+}),$$

and so

$$\begin{aligned}
\|M_{\Psi_{\beta,\mathcal{E}}}(\{a_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^n_+})\|^2_{\mathbb{H}^2_n(\gamma,\mathcal{E})} &= \Big\| \sum_{\boldsymbol{k}\in\mathbb{Z}^n_+} \Big(\sqrt{\frac{\rho_1(\boldsymbol{k})}{\gamma_{|\boldsymbol{k}|}}} a_{\boldsymbol{k}}\Big) z^{\boldsymbol{k}} \Big\|^2_{\mathbb{H}^2_n(\gamma,\mathcal{E})} \\
&= \sum_{\boldsymbol{k}\in\mathbb{Z}^n_+} \frac{\rho_1(\boldsymbol{k})}{\gamma_{|\boldsymbol{k}|}} \|a_{\boldsymbol{k}}\|^2_{\mathcal{E}} \|z^{\boldsymbol{k}}\|^2_{\mathbb{H}^2_n(\gamma,\mathcal{E})} \\
&= \sum_{\boldsymbol{k}\in\mathbb{Z}^n_+} \frac{\rho_1(\boldsymbol{k})}{\gamma_{|\boldsymbol{k}|}} \|a_{\boldsymbol{k}}\|^2_{\mathcal{E}} \frac{\gamma_{|\boldsymbol{k}|}}{\rho_1(\boldsymbol{k})} \\
&= \|\{a_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^n_+}\|^2_{l^2(\mathbb{Z}^n_+,\mathcal{E})}.
\end{aligned}$$

We now relate the idea of universal multipliers to uniqueness of factorizations of multipliers in the context of Theorem 4.2.

Theorem 4.3. *Let $\mathcal{E}$ and $\mathcal{E}_*$ be Hilbert spaces, β be a weight sequence, and let*

$$\Theta \in \mathcal{SM}(H^2_n(\mathcal{E}), \mathbb{H}^2_n(\beta, \mathcal{E}_*)).$$

If Θ is a K_β-inner multiplier then there exists a unique K-inner multiplier

$$\tilde{\Theta} \in \mathcal{SM}(H^2_n(\mathcal{E}), H^2_n(l^2(\mathbb{Z}^n_+, \mathcal{E}_*)),$$

such that

$$\Theta(\boldsymbol{z}) = \Psi_{\beta,\mathcal{E}_*}(\boldsymbol{z})\tilde{\Theta}(\boldsymbol{z}) \qquad (\boldsymbol{z} \in \mathbb{B}^n).$$

Proof. If $\Theta \in \mathcal{SM}(H^2_n(\mathcal{E}), \mathbb{H}^2_n(\beta, \mathcal{E}_*))$, then by Theorem 4.2, we have

$$\Theta = \Psi_{\beta,\mathcal{E}_*}\tilde{\Theta}.$$

for some $\tilde{\Theta} \in \mathcal{SM}(H^2_n(\mathcal{E}), \mathbb{H}^2_n(l^2(\mathbb{Z}^n_+, \mathcal{E}_*))$. Now let Θ be K_β-inner. We show that $\tilde{\Theta}$ is K-inner. Let $\eta \in \mathcal{E}$, and let

$$\tilde{\Theta}\eta = f \oplus g \in \ker M_{\Psi_{\beta,\mathcal{E}_*}} \oplus (\ker M_{\Psi_{\beta,\mathcal{E}_*}})^\perp.$$

Since

$$\|\eta\|_{\mathcal{E}} = \|M_\Theta \eta\|_{\mathbb{H}^2_n(\beta,\mathcal{E}_*)} = \|M_{\Psi_{\beta,\mathcal{E}_*}} M_{\tilde{\Theta}}\eta\|_{\mathbb{H}^2_n(\beta,\mathcal{E}_*)}$$

and $M_{\Psi_{\beta,\mathcal{E}_*}}$ is a co-isometry, we see that

$$\begin{aligned}\|\eta\|_{\mathcal{E}} &= \|M_{\Psi_{\beta,\mathcal{E}_*}} M_{\tilde{\Theta}}\eta\|_{\mathbb{H}^2_n(\beta,\mathcal{E}_*)}\\ &= \|M_{\Psi_{\beta,\mathcal{E}_*}} g\|_{\mathbb{H}^2_n(\beta,\mathcal{E}_*)}\\ &= \|g\|_{H^2_n(l^2(\mathbb{Z}^n_+,\mathcal{E}_*))}\\ &\leq \|\tilde{\Theta}\eta\|_{H^2_n(l^2(\mathbb{Z}^n_+,\mathcal{E}_*))}\\ &\leq \|\eta\|_{\mathcal{E}}.\end{aligned}$$

It now follows that $\|\tilde{\Theta}\eta\|_{H^2_n(l^2(\mathbb{Z}^n_+,\mathcal{E}_*))} = \|\eta\|_{\mathcal{E}}$ for all $\eta \in \mathcal{E}$ and

$$\tilde{\Theta}\mathcal{E} \subseteq \Big(\ker M_{\Psi_{\beta,\mathcal{E}_*}}\Big)^{\perp}.$$

This readily shows that

$$M^*_{\Psi_{\beta,\mathcal{E}_*}} M_{\Psi_{\beta,\mathcal{E}_*}}|_{\tilde{\Theta}\mathcal{E}} = I.$$

Therefore, for $\eta, \zeta \in \mathcal{E}$ and $\boldsymbol{k} \in \mathbb{N}^n$, we see that

$$\begin{aligned}\langle \tilde{\Theta}\eta, z^{\boldsymbol{k}}\tilde{\Theta}\zeta\rangle_{H^2_n(l^2(\mathbb{Z}^n_+,\mathcal{E}_*))} &= \langle M^*_{\Psi_{\beta,\mathcal{E}_*}} M_{\Psi_{\beta,\mathcal{E}_*}}\tilde{\Theta}\eta, \tilde{\Theta}z^{\boldsymbol{k}}\zeta\rangle_{H^2_n(l^2(\mathbb{Z}^n_+,\mathcal{E}_*))}\\ &= \langle \Psi_{\beta,\mathcal{E}_*}\tilde{\Theta}\eta, \Psi_{\beta,\mathcal{E}_*}\tilde{\Theta}z^{\boldsymbol{k}}\zeta\rangle_{\mathbb{H}^2_n(\beta,\mathcal{E}_*))}\\ &= \langle \Theta\eta, \Theta z^{\boldsymbol{k}}\zeta\rangle_{\mathbb{H}^2_n(\beta,\mathcal{E}_*)}\\ &= \langle \Theta\eta, z^{\boldsymbol{k}}\Theta\zeta\rangle_{\mathbb{H}^2_n(\beta,\mathcal{E}_*)},\end{aligned}$$

and hence the orthogonality condition of K_β-inner multiplier Θ implies that of $\tilde{\Theta}$. Finally, since

$$M_{\tilde{\Theta}}(z^{\boldsymbol{k}}\eta) = z^{\boldsymbol{k}}\tilde{\Theta}\eta = z^{\boldsymbol{k}} M^*_{\Psi_{\beta,\mathcal{E}_*}} M_{\Psi_{\beta,\mathcal{E}_*}}\tilde{\Theta}\eta = z^{\boldsymbol{k}} M^*_{\Psi_{\beta,\mathcal{E}_*}}\Theta\eta,$$

for all $\eta \in \mathcal{E}$ and $\boldsymbol{k} \in \mathbb{Z}^n_+$, it follows that $\tilde{\Theta}$ is unique. This completes the proof of the theorem. ■

In the particular case $n = 1$, all the results obtained so far in this section are due to Ball and Bolotnikov [7].

The discussion to this point motivates us to define wandering subspaces of bounded linear operators. The notion of a wandering subspace was introduced by Halmos [11] in the context of invariant subspaces of shifts on vector-valued Hardy spaces. Let T be an n-tuple of commuting operators on $\mathcal{H}$, and let $\mathcal{W}$ be a closed subspace of $\mathcal{H}$. If

$$\mathcal{W} \perp T^{\boldsymbol{k}}\mathcal{W},$$

for all $\boldsymbol{k} \in \mathbb{N}^n$, then $\mathcal{W}$ is called a *wandering subspace* for T. We say that $\mathcal{W}$ is a *generating wandering subspace* for T if in addition

$$\mathcal{H} = \overline{\text{span}}\{T^{\boldsymbol{k}}\mathcal{W} : \boldsymbol{k} \in \mathbb{Z}^n_+\}.$$

Here, however, we aim at parameterizing wandering subspaces for $M_z = (M_{z_1}, \ldots, M_{z_n})$ on $\mathbb{H}^2_n(\beta, \mathcal{E}_*)$. Note, by virtue of (4.1) and (4.2), that the tuple of multiplication operator M_z

defines a pure row contraction on $\mathbb{H}_n^2(\beta, \mathcal{E}_*)$. Let $\mathcal{W}$ be a wandering subspace for M_z on $\mathbb{H}_n^2(\beta, \mathcal{E}_*)$. Clearly

$$\bigvee_{\boldsymbol{k} \in \mathbb{Z}_+^n} z^{\boldsymbol{k}} \mathcal{W},$$

is a joint M_z-invariant subspace of $\mathbb{H}_n^2(\beta, \mathcal{E}_*)$. Then there exist a Hilbert space $\mathcal{E}$ and a partial isometric multiplier $\Theta \in \mathcal{M}(H_n^2(\mathcal{E}), \mathbb{H}_n^2(\beta, \mathcal{E}_*))$ such that

$$\bigvee_{\boldsymbol{k} \in \mathbb{Z}_+^n} z^{\boldsymbol{k}} \mathcal{W} = \Theta H_n^2(\mathcal{E}).$$

Moreover, if

$$\mathcal{F} = \{\eta \in \mathcal{E} : M_\Theta^* M_\Theta \eta = \eta\} \subseteq \mathcal{E},$$

then the wandering subspace $\mathcal{W}$ and the multiplier Θ are related as follows (see Theorem 6.6, [8]):

$$\mathcal{W} = \Theta \mathcal{F},$$

and

$$\Theta|_{H_n^2(\mathcal{F})} \in \mathcal{SM}(H_n^2(\mathcal{F}), \mathbb{H}_n^2(\beta, \mathcal{E}_*))$$

is a K_β-inner function. Now we apply Theorem 4.3 to the K_β-inner function $\Theta|_{H_n^2(\mathcal{F})}$ and get that

$$\Theta|_{H_n^2 \mathcal{F}} = \Psi_{\beta, \mathcal{E}_*} \tilde{\Theta},$$

where $\tilde{\Theta} \in \mathcal{SM}(H_n^2(\mathcal{F}), H_n^2(l^2(\mathbb{Z}_+^n, \mathcal{E}_*)))$ is the unique K-inner multiplier. In particular,

$$\tilde{\mathcal{W}} := \tilde{\Theta} \mathcal{F},$$

is a wandering subspace for M_z on $H_n^2(l^2(\mathbb{N}^n, \mathcal{E}_*))$, and so

$$\mathcal{W} = \Psi_{\beta, \mathcal{E}_*} \tilde{\Theta} \mathcal{F} = \Psi_{\beta, \mathcal{E}_*} \tilde{\mathcal{W}}.$$

This yields the following parametrization of a wandering subspace for M_z on $\mathbb{H}_n^2(\beta, \mathcal{E}_*)$.

Theorem 4.4. *If $\mathcal{W}$ is a wandering subspace for M_z on $\mathbb{H}_n^2(\beta, \mathcal{E}_*)$, then there exists a wandering subspace $\tilde{\mathcal{W}}$ for M_z on $H_n^2(l^2(\mathbb{Z}_+^n, \mathcal{E}_*))$ such that*

$$\mathcal{W} = \Psi_{\beta, \mathcal{E}_*} \tilde{\mathcal{W}},$$

where $\Psi_{\beta, \mathcal{E}_*}$ is the universal multiplier.

The above parametrizations of wandering subspaces is significantly different from that of Eschmeier [10] and Olofsson [17].

5. Factorizations and representations of Characteristic Functions

We continue our study of pure m-hypercontractions by focusing on the universal multipliers $\Psi_{\beta,\mathcal{H}}$ and relate this idea to the notion of the transfer functions on $\mathbb{B}^n$. Here we follow the notation introduced in Section 4.

Fix $m > 1$ and a weight sequence $\beta(m) = \{\beta_j(m)\}$ as

$$\beta_j(m) = \binom{m+j-1}{j}^{-1},$$

for all $j \in \mathbb{Z}_+$. Then the corresponding weight sequence $\gamma(m) = \{\gamma_j(m)\}$ is given by

$$\begin{aligned}\gamma_j(m)^{-1} &= \frac{1}{\beta_j(m)} - \frac{1}{\beta_{j-1}(m)}\\ &= \frac{(m+j-1)!}{j!(m-1)!} - \frac{(m+j-2)!}{(j-1)!(m-1)!}\\ &= \frac{(m+j-2)!}{j!(m-2)!},\end{aligned}$$

that is

$$\gamma_j(m) = \binom{m+j-2}{j}^{-1},$$

for all $j \geq 1$. Then for a Hilbert space $\mathcal{F}$, one finds that

$$\mathbb{H}^2_n(\beta(m), \mathcal{F}) = \mathbb{H}_m(\mathbb{B}^n, \mathcal{F}) \quad \text{and} \quad \mathbb{H}^2_n(\gamma(m), \mathcal{F}) = \mathbb{H}_{m-1}(\mathbb{B}^n, \mathcal{F}). \tag{5.1}$$

Now let T be a pure m-hypercontraction on $\mathcal{H}$, and let $(\mathcal{E}, B, D)$ be a characteristic triple of T. Then Φ_T, the characteristic function of T corresponding to $(\mathcal{E}, B, D)$, defined by

$$\Phi_T(\boldsymbol{z}) = \Big(\sum_{\boldsymbol{k}\in\mathbb{Z}^n_+} \sqrt{\rho_{m-1}(\boldsymbol{k})} D_{\boldsymbol{k}} z^{\boldsymbol{k}}\Big) + D_{m,T^*}(I_{\mathcal{H}} - ZT^*)^{-m} ZB \qquad (\boldsymbol{z} \in \mathbb{B}^n),$$

is a $\mathcal{B}(\mathcal{E}, \mathcal{D}_{m,T^*})$-valued analytic function on $\mathbb{B}^n$. Moreover

$$\Phi_T \in \mathcal{M}(H^2_n(\mathcal{E}), \mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})),$$

is a partially isometric multiplier (see Theorem 3.1). Now, in view of (5.1), Theorem 4.2 implies that

$$\Phi_T = \Psi_{\beta(m),\mathcal{D}_{m,T^*}} \tilde{\Phi}_T,$$

for some contractive multiplier $\tilde{\Phi}_T \in \mathcal{M}(H^2_n(\mathcal{E}), H^2_n(l^2(\mathbb{Z}^n_+, \mathcal{D}_{m,T^*})))$. Here

$$\Psi_{\beta(m),\mathcal{D}_{m,T^*}} \in \mathcal{M}(H^2_n(l^2(\mathbb{Z}^n_+, \mathcal{D}_{m,T^*})), \mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})),$$

is the universal multiplier defined by

$$\Psi_{\beta(m),\mathcal{D}_{m,T^*}}(\boldsymbol{z}) = \Big[\cdots \sqrt{\frac{\rho_1(\boldsymbol{k})}{\gamma_{|\boldsymbol{k}|}(m)}} z^{\boldsymbol{k}} I_{\mathcal{D}_{m,T^*}} \cdots\Big]_{\boldsymbol{k}\in\mathbb{Z}^n_+},$$

for all $\boldsymbol{z} \in \mathbb{B}^n$. However, in our particular situation

$$\gamma_{|\boldsymbol{k}|}(m) = \binom{m + |\boldsymbol{k}| - 2}{|\boldsymbol{k}|},$$

and hence

$$\sqrt{\frac{\rho_1(\boldsymbol{k})}{\gamma_{|\boldsymbol{k}|}(m)}} = \sqrt{\rho_{m-1}(\boldsymbol{k})},$$

for all $\boldsymbol{k} \in \mathbb{Z}_+^n$. Then the universal multiplier is given by

$$\Psi_{\beta(m),\mathcal{D}_{m,T^*}}(\boldsymbol{z}) = \left[\cdots \sqrt{\rho_{m-1}(\boldsymbol{k})} z^{\boldsymbol{k}} I_{\mathcal{D}_{m,T^*}} \cdots \right]_{\boldsymbol{k} \in \mathbb{Z}_+^n}. \tag{5.2}$$

Now we proceed to compute an explicit representation of $\tilde{\Phi}_T$. To this end, we first recall that

$$U = \begin{bmatrix} T^* & B \\ C_{m,T} & D \end{bmatrix} : \mathcal{H} \oplus \mathcal{E} \to \mathcal{H}^n \oplus l^2(\mathbb{Z}_+^n, \mathcal{D}_{m,T^*})$$

is unitary (see Theorem 2.1). We claim that $\tilde{\Phi}_T$ is the transfer of the unitary U (see [3]), that is,

$$\tilde{\Phi}_T(\boldsymbol{z}) = D + C_{m,T}(I_{\mathcal{H}} - ZT^*)^{-1} ZB \qquad (\boldsymbol{z} \in \mathbb{B}^n).$$

Indeed, first note that $\tilde{\Phi}_T \in \mathcal{M}(H_n^2(\mathcal{E}), H_n^2(l^2(\mathbb{Z}_+^n, \mathcal{D}_{m,T^*})))$ (cf. [3]) and

$$\begin{aligned}
\tilde{\Phi}_T(z) &= D + C_{m,T}(I_{\mathcal{H}} - ZT^*)^{-1} ZB \\
&= D + C_{m,T} \sum_{i=1}^{n} \big(\sum_{\boldsymbol{l} \in \mathbb{Z}_+^n} \rho_1(\boldsymbol{l}) z^{\boldsymbol{l}} T^{*\boldsymbol{l}} \big) z_i B_i \\
&= D + \sum_{i=1}^{n} \sum_{\boldsymbol{l} \in \mathbb{Z}_+^n} \big(\rho_1(\boldsymbol{l}) C_{m,T} T^{*\boldsymbol{l}} B_i \big) z^{\boldsymbol{l} + \boldsymbol{e}_i},
\end{aligned}$$

for all $\boldsymbol{z} \in \mathbb{B}^n$, where

$$B = \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} : \mathcal{E} \to \mathcal{H}^n,$$

and $\boldsymbol{e}_i \in \mathbb{Z}_+^n$ has a 1 in the i-th position and 0 elsewhere, $i = 1, \ldots, n$. Then, by applying the matrix representation of $C_{m,T}$ (see (2.6)), we have

$$\tilde{\Phi}_T(\boldsymbol{z}) = \begin{bmatrix} \vdots \\ D_{\boldsymbol{k}} + \sum_{i=1}^{n} \sum_{\boldsymbol{l} \in \mathbb{Z}_+^n} \big(\sqrt{\rho_{m-1}(\boldsymbol{k})} \rho_1(\boldsymbol{l}) D_{m,T^*} T^{*(\boldsymbol{k}+\boldsymbol{l})} B_i \big) z^{\boldsymbol{l} + \boldsymbol{e}_i} \\ \vdots \end{bmatrix}_{\boldsymbol{k} \in \mathbb{Z}_+^n},$$

and consequently, by (5.2), we have

$$\Psi_{\beta(m),\mathcal{D}_{m,T^*}}(\boldsymbol{z})\tilde{\Phi}_T(\boldsymbol{z}) = \sum_{\boldsymbol{k}\in\mathbb{Z}_+^n}\Big(\sqrt{\rho_{m-1}(\boldsymbol{k})}D_{\boldsymbol{k}} + \sum_{i=1}^n\sum_{\boldsymbol{l}\in\mathbb{Z}_+^n}\big(\rho_{m-1}(\boldsymbol{k})\rho_1(\boldsymbol{l})D_{m,T^*}T^{*(\boldsymbol{k}+\boldsymbol{l})}B_i\big)z^{\boldsymbol{l}+\boldsymbol{e}_i}\Big)z^{\boldsymbol{k}}.$$

Also note that

$$\begin{aligned}
D_{m,T^*}(I_{\mathcal{H}}-ZT^*)^{-m}ZB &= D_{m,T^*}(I_{\mathcal{H}}-ZT^*)^{-(m-1)}(I_{\mathcal{H}}-ZT^*)^{-1}ZB\\
&= D_{m,T^*}\Big(\sum_{\boldsymbol{k}\in\mathbb{Z}_+^n}\rho_{m-1}(\boldsymbol{k})T^{*\boldsymbol{k}}z^{\boldsymbol{k}}\Big)\Big(\sum_{\boldsymbol{l}\in\mathbb{Z}_+^n}\rho_1(\boldsymbol{l})T^{*\boldsymbol{l}}z^{\boldsymbol{l}}\Big)ZB\\
&= D_{m,T^*}\sum_{i=1}^n z_i\Big(\sum_{\boldsymbol{k}\in\mathbb{Z}_+^n}\rho_{m-1}(\boldsymbol{k})T^{*\boldsymbol{k}}z^{\boldsymbol{k}}\Big)\Big(\sum_{\boldsymbol{l}\in\mathbb{Z}_+^n}\rho_1(\boldsymbol{l})T^{*\boldsymbol{l}}z^{\boldsymbol{l}}\Big)B_i\\
&= \sum_{\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}_+^n}\sum_{i=1}^n\Big(\rho_{m-1}(\boldsymbol{k})\rho_1(\boldsymbol{l})D_{m,T^*}T^{*(\boldsymbol{k}+\boldsymbol{l})}B_i\Big)z^{\boldsymbol{k}+\boldsymbol{l}+\boldsymbol{e}_i}.
\end{aligned}$$

From this it readily follows that

$$\begin{aligned}
\Psi_{\beta(m),\mathcal{D}_{m,T^*}}(\boldsymbol{z})\tilde{\Phi}_T(\boldsymbol{z}) &= \Big(\sum_{\boldsymbol{k}\in\mathbb{Z}_+^n}\sqrt{\rho_{m-1}(\boldsymbol{k})}D_{\boldsymbol{k}}z^{\boldsymbol{k}}\Big) + D_{m,T^*}(I_{\mathcal{H}}-ZT^*)^{-m}ZB\\
&= \Phi_T(\boldsymbol{z}),
\end{aligned}$$

for all $\boldsymbol{z}\in\mathbb{B}^n$. This leads to the following theorem on explicit representation of $\tilde{\Phi}_T$:

Theorem 5.1. *Let $m\geq 1$, T be a pure m-hypercontraction on a Hilbert space $\mathcal{H}$, and let $(\mathcal{E},B,D)$ be a characteristic triple of T. If Φ_T is the characteristic function of T corresponding to $(\mathcal{E},B,D)$, then*

$$\Phi_T(\boldsymbol{z}) = \Psi_{\beta(m),\mathcal{D}_{m,T^*}}(\boldsymbol{z})\tilde{\Phi}_T(\boldsymbol{z}) \qquad (\boldsymbol{z}\in\mathbb{B}^n),$$

where

$$\tilde{\Phi}_T(z) = D + C_{m,T}(I_{\mathcal{H}}-ZT^*)^{-1}ZB \qquad (\boldsymbol{z}\in\mathbb{B}^n),$$

is the transfer function of the canonical unitary matrix

$$\begin{bmatrix} T^* & B\\ C_{m,T} & D\end{bmatrix} : \mathcal{H}\oplus\mathcal{E}\to\mathcal{H}^n\oplus l^2(\mathbb{Z}_+^n,\mathcal{D}_{m,T^*})$$

corresponding to the characteristic triple $(\mathcal{E},B,D)$ of T, and

$$\Psi_{\beta(m),\mathcal{D}_{m,T^*}}(\boldsymbol{z}) = \begin{cases}\begin{bmatrix}\cdots & \sqrt{\rho_{m-1}(\boldsymbol{k})}z^{\boldsymbol{k}}I_{\mathcal{D}_{m,T^*}} & \cdots\end{bmatrix}_{\boldsymbol{k}\in\mathbb{Z}_+^n} & \text{if } m\geq 2\\ \begin{bmatrix} I_{\mathcal{D}_{1,T^*}} & 0 & 0 & \cdots\end{bmatrix} & \text{if } m=1,\end{cases}$$

for all $\boldsymbol{z}\in\mathbb{B}^n$.

Proof. It remains only to prove the special case $m=1$. Let T be a pure 1-hypercontraction, and let $(\mathcal{E},B,D)$ be a characteristic triple of T. Then (2.5) implies that

$$C_{1,T}(h) = (D_{1,T^*}h,0,0,\ldots) \qquad (h\in\mathcal{H}),$$

and so

$$\Phi_T = D_0 + D_{1,T^*}(I_{\mathcal{H}} - ZT^*)^{-1}ZB,$$

for all $\boldsymbol{z} \in \mathbb{B}^n$. It now easily follows that

$$\Phi_T = \Psi_{\beta(1),\mathcal{D}_{1,T^*}}(\boldsymbol{z})\tilde{\Phi}_T(\boldsymbol{z}) \qquad (\boldsymbol{z} \in \mathbb{B}^n).$$

■

We will refer

$$\tilde{\Phi}_T \in \mathcal{M}(H^2_n(\mathcal{E}), \mathbb{H}_m(l^2(\mathbb{Z}^n_+, \mathcal{D}_{m,T^*})),$$

as the *canonical transfer function* of T corresponding to the characteristic triple $(\mathcal{E}, B, D)$.

6. Hypercontractions and row-contractions

The present theory of pure m-hypercontractions leads to many interesting questions of analytic models, such as any possible relationships between characteristic functions or canonical transfer functions of m'-hypercontractions, $1 \leq m' < m$. Here we address this issue. Also we compare the ideas of characteristic functions of pure m-hypercontractions and characteristic functions of pure row contractions.

First, we examine our construction of characteristic triples for pure 1-hypercontractions. Before doing so we recall that the *characteristic function* [9] of a commuting row contraction (that is, 1-hypercontraction) $T = (T_1, \ldots, T_n)$ on a Hilbert space $\mathcal{H}$ is the operator-valued analytic function

$$\Theta_T(\boldsymbol{z}) = [-T + D_{1,T^*}(I_{\mathcal{H}} - ZT^*)^{-1}ZD_T]|_{\mathcal{D}_T} \in \mathcal{B}(\mathcal{D}_T, \mathcal{D}_{1,T^*}) \qquad (\boldsymbol{z} \in \mathbb{B}^n),$$

where $D_T = (I_{\mathcal{H}^n} - T^*T)^{\frac{1}{2}}$ and $\mathcal{D}_T = \overline{\mathrm{ran}} D_T$. Observe also that Θ_T is the transfer function corresponding to the unitary (colligation) matrix

$$\begin{bmatrix} T^* & D_T \\ D_{1,T^*} & -T \end{bmatrix} : \mathcal{H} \oplus \mathcal{D}_T \to \mathcal{H}^n \oplus \mathcal{D}_{1,T^*},$$

and $\Theta_T \in \mathcal{M}(H^2_n(\mathcal{D}_T), H^2_n(\mathcal{D}_{1,T^*}))$ (cf. [9]). In the following, we shall identify $\mathcal{D}_{1,T^*}$ with

$$\mathcal{D}_{1,T^*} \oplus \{0\} \oplus \{0\} \oplus \cdots \subset l^2(\mathbb{Z}^n_+, \mathcal{D}_{1,T^*}),$$

and view $\Theta_T \in \mathcal{M}(H^2_n(\mathcal{D}_T), H^2_n(\mathcal{D}_{1,T^*} \oplus \{0\} \oplus \{0\} \oplus \cdots))$.

Theorem 6.1. *Let T be a pure row contraction on $\mathcal{H}$. Then there exists a characteristic triple $(\mathcal{E}, B, D)$ of T such that $\mathcal{D}_T \subseteq \mathcal{E}$ and*

$$\Theta_T(\boldsymbol{z}) = \tilde{\Phi}_T(\boldsymbol{z})|_{\mathcal{D}_T} \qquad (\boldsymbol{z} \in \mathbb{B}^n),$$

where $\tilde{\Phi}_T$, defined by

$$\tilde{\Phi}_T(\boldsymbol{z}) = D + C_{1,T}(I_{\mathcal{H}} - ZT^*)^{-1}ZB \qquad (\boldsymbol{z} \in \mathbb{B}^n),$$

and Θ_T are the canonical transfer function corresponding to $(\mathcal{E}, B, D)$ and the characteristic function of T, respectively.

Proof. Let T be a pure row contraction (that is, pure 1-hypercontraction). Set
$$\mathcal{E} := \mathcal{D}_T \oplus \tilde{\mathcal{E}},$$
where $\tilde{\mathcal{E}} = l^2(\mathbb{Z}_+^n, \mathcal{D}_{1,T^*}) \ominus (\mathcal{D}_{1,T^*}, 0, \cdots)$. Define $B = [D_T, 0] : \mathcal{E} \to \mathcal{H}^n$ by
$$B(f, \{\alpha_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}_+^n}) = D_T f,$$
and $D : \mathcal{E} \to l^2(\mathbb{Z}_+^n, \mathcal{D}_{1,T^*})$ by
$$\Big(D(f, \{\alpha_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}_+^n})\Big)(\boldsymbol{l}) = \begin{cases} -Tf & \text{if } \boldsymbol{l} = \boldsymbol{0} \\ \alpha_{\boldsymbol{l}} & \text{otherwise} \end{cases}$$
for all $f \in \mathcal{D}_T$ and $\{\alpha_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}_+^n} \in \tilde{\mathcal{E}}$. Finally, define $C_{1,T} : \mathcal{H} \to l^2(\mathbb{Z}_+^n, \mathcal{D}_{1,T^*})$ by
$$C_{1,T}h = (D_{1,T^*}h, 0, \cdots) \quad (h \in \mathcal{H}).$$
It is obvious that
$$TT^* + C_{1,T}^* C_{1,T} = I_{\mathcal{H}},$$
and
$$\begin{bmatrix} T^* & B \\ C_{1,T} & D \end{bmatrix} : \mathcal{H} \oplus \mathcal{E}_1 \to \mathcal{H}^n \oplus l^2(\mathbb{Z}_+^n, \mathcal{D}_{1,T^*}),$$
is unitary, which implies that $(\mathcal{E}, B, D)$ is a characteristic triple of the 1-hypercontraction T. The canonical transfer function corresponding to $(\mathcal{E}, B, D)$ is given by
$$\tilde{\Phi}_T(\boldsymbol{z}) = D + C_{1,T}(I_{\mathcal{H}} - ZT^*)^{-1} ZB \qquad (\boldsymbol{z} \in \mathbb{B}^n).$$
Then it readily follows that $\Theta_T(\boldsymbol{z}) = \tilde{\Phi}_T(\boldsymbol{z})|_{\mathcal{D}_T}$ for all $\boldsymbol{z} \in \mathbb{B}^n$. This completes the proof of the theorem. ∎

We refer to the characteristic triple constructed above for a pure 1-hypercontraction as the *canonical characteristic triple*.

Now let $1 \le m_1 < m_2$ and let T be a pure m_2-hypercontraction on $\mathcal{H}$. Then T is also a pure m_1-hypercontraction. Suppose that $(\mathcal{E}_i, B_i, D_i)$ is a characteristic triple of the m_i-hypercontraction T, $i = 1, 2$. Then
$$U_i = \begin{bmatrix} T^* & B_i \\ C_{m_i,T} & D_i \end{bmatrix} : \mathcal{H} \oplus \mathcal{E}_i \to \mathcal{H}^n \oplus l^2(\mathbb{Z}_+^n, \mathcal{D}_{m_i,T^*}),$$
is the unitary operator corresponding to the m_i-hypercontraction T, $i = 1, 2$. For simplicity of notation, we denote $\tilde{\Phi}_{T,m_i}$ the canonical transfer function corresponding to $(\mathcal{E}_i, B_i, D_i)$, $i = 1, 2$. Since (see (2.7))
$$C_{m_i,T}^* C_{m_i,T} = I_{\mathcal{H}} - TT^* \qquad (i = 1, 2),$$
we have
$$C_{m_1,T}^* C_{m_1,T} = C_{m_2,T}^* C_{m_2,T}.$$
Also, according to (3.3), we have
$$B_1 B_1^* = B_2 B_2^*$$
and
$$D_i B_i^* = -C_{m_i,T} T,$$

for $i = 1, 2$. It now follows by Douglas' range inclusion theorem that

$$YC_{m_2,T} = C_{m_1,T} \quad \text{and} \quad XB_1^* = B_2^*,$$

for some isometry $Y \in \mathcal{B}(\overline{\text{ran}}\ C_{m_2,T}, l^2(\mathbb{Z}_+^n, \mathcal{D}_{m_1,T^*}))$ and unitary $X \in \mathcal{B}(\overline{\text{ran}}\ B_1^*, \overline{\text{ran}}\ B_2^*)$. Thus

$$D_1 B_1^* = -C_{m_1,T} T = -YC_{m_2,T} T = YD_2 B_2^* = YD_2 X B_1^*,$$

and so

$$D_1|_{(\ker B_1)^\perp} = YD_2X.$$

This and the definition of $\tilde{\Phi}_{T,m_i}$, $i = 1, 2$, gives

$$\begin{aligned} \tilde{\Phi}_{T,m_1}(\boldsymbol{z})|_{(\ker B_1)^\perp} &= \left[D_1 + C_{m_1,T}(I_{\mathcal{H}} - ZT^*)^{-1} Z B_1\right]|_{(\ker B_1)^\perp} \\ &= YD_2X + YC_{m_2,T}(I_{\mathcal{H}} - ZT^*)^{-1} Z B_2 X \\ &= Y\tilde{\Phi}_{T,m_2}(\boldsymbol{z})X. \end{aligned}$$

This establishes the following relationship between canonical transfer functions:

THEOREM 6.2. *Let $1 \le m_1 < m_2$, T be a pure m_2-hypercontraction on $\mathcal{H}$, and let $(\mathcal{E}_i, B_i, D_i)$ be characteristic triple of the m_i-hypercontraction T, $i = 1, 2$. Then there exist an isometry $Y \in \mathcal{B}(\overline{ran}\ C_{m_2,T}, l^2(\mathbb{Z}_+^n, \mathcal{D}_{m_1,T^*}))$ and a unitary $X \in \mathcal{B}(\overline{ran}\ B_1^*, \overline{ran}\ B_2^*)$ such that*

$$\tilde{\Phi}_{T,m_1}(\boldsymbol{z})|_{(\ker B_1)^\perp} = Y\tilde{\Phi}_{T,m_2}(\boldsymbol{z})X \qquad (\boldsymbol{z} \in \mathbb{B}^n),$$

where $\tilde{\Phi}_{T,m_i}$ is the canonical transfer function corresponding to the characteristic triple $(\mathcal{E}_i, B_i, D_i)$, $i = 1, 2$.

REMARK 6.3. *Let $\mathcal{F}$, $\mathcal{F}_*$, $\mathcal{E}$ and $\mathcal{E}_*$ be Hilbert spaces, and let*

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathcal{F} \oplus \mathcal{E} \to \mathcal{F}_* \oplus \mathcal{E}_*,$$

be a unitary. Suppose that Φ is the transfer function corresponding to U, that is

$$\Phi(\boldsymbol{z}) = D + C(I - ZA)^{-1} ZB \qquad (\boldsymbol{z} \in \mathbb{B}^n).$$

Then $\Phi|_{(\ker B)^\perp}$ is the purely contractive part of the contractive operator-valued analytic function Φ on $\mathbb{B}^n$ in the sense of Sz.-Nagy and Foias [15, Chapter V, Proposition 2.1]. *This follows from the observation that the maximal subspace of $\mathcal{E}$ where D is an isometry is $\ker B$ and $D|_{\ker B} : \ker B \to \ker C^*$ is a unitary. Moreover, $\Phi|_{(\ker B)^\perp}$ is the transfer function of the unitary*

$$\begin{bmatrix} A & B|_{(\ker B)^\perp} \\ C & D|_{(\ker B)^\perp} \end{bmatrix} : \mathcal{F} \oplus (\ker B)^\perp \to \mathcal{F}_* \oplus \overline{ran} C.$$

From this point of view, $\tilde{\Phi}_{T,m_1}(\boldsymbol{z})|_{(\ker B_1)^\perp}$, $\boldsymbol{z} \in \mathbb{B}^n$, in the conclusion of Theorem 6.2 is the purely contractive part of $\tilde{\Phi}_{T,m_1}$. Moreover, $ranX = (\ker B_2)^\perp$ implies that $Y\tilde{\Phi}_{T,m_1}(.)X$ coincides with the purely contactive part of $\tilde{\Phi}_{T,m_1}$. Therefore Theorem 6.2 implies that the purely contractive part of $\tilde{\Phi}_{T,m_1}$ coincides with the purely contractive part of $\tilde{\Phi}_{T,m_2}$.

We continue with the hypothesis that T is a pure m-hypercontraction, $m > 1$. Let $(\mathcal{E}_m, B_m, D_m)$ be a characteristic triple of the pure m-hypercontraction T and $\tilde{\Phi}_{T,m}$ be the corresponding canonical transfer function. Since T is also a pure 1-hypercontraction, consider the canonical characteristic triple $(\mathcal{E}, B, D)$ of T as obtained in the proof of Theorem 6.1. Let $\tilde{\Phi}_T$ be the canonical transfer function corresponding to $(\mathcal{E}, B, D)$. Then by Theorem 6.2,

$$\tilde{\Phi}_T(\boldsymbol{z})|_{(\ker B)^\perp} = Y \tilde{\Phi}_{T,m}(\boldsymbol{z}) X \qquad (\boldsymbol{z} \in \mathbb{B}^n),$$

for some isometry $Y \in \mathcal{B}(\overline{\text{ran}}\ C_{m,T}, l^2(\mathbb{Z}_+^n, \mathcal{D}_{1,T^*}))$ and unitary $X \in \mathcal{B}(\overline{\text{ran}}\ B^*, \overline{\text{ran}}\ B_m^*)$. Moreover (see the construction of B in the proof of Theorem 6.1)

$$(\ker B)^\perp = \mathcal{D}_T,$$

and hence by Theorem 6.1, it follows that

$$\Theta_T(\boldsymbol{z}) = Y \tilde{\Phi}_{T,m}(\boldsymbol{z}) X \qquad (\boldsymbol{z} \in \mathbb{B}^n).$$

Therefore, we have the following theorem:

THEOREM 6.4. *Let $m \geq 2$, T be a pure m-hypercontraction on $\mathcal{H}$, and let $(\mathcal{E}_m, B_m, D_m)$ be a characteristic triple of T. Then there exist an isometry $Y \in \mathcal{B}(\overline{ran}\ C_{m,T}, l^2(\mathbb{Z}_+^n, \mathcal{D}_{1,T^*}))$ and a unitary $X \in \mathcal{B}(\mathcal{D}_T, \overline{ran}\ B_m^*)$ such that*

$$\Theta_T(\boldsymbol{z}) = Y \tilde{\Phi}_{T,m}(\boldsymbol{z}) X \qquad (\boldsymbol{z} \in \mathbb{B}^n),$$

where Θ_T and $\tilde{\Phi}_{m,T}$ denote the characteristic function of the row contraction T and the canonical transfer function of T corresponding to the characteristic triple $(\mathcal{E}_m, B_m, D_m)$, respectively.

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